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# Solid Slider Bearings Lubricated by Their Own Melting or Sublimation

*We study the steady-state hydrodynamic lubrication of a solid-solid sliding bearing with spontaneous fusion or sublimation of the solid slider substance. Lubrication is sustained by the fluid film produced by fusion or sublimation. Our analysis extends the recent literature on liquid film lubrication of a melting solid slider to the interesting case of gaseous film lubrication of a sublimating solid slider. The results are in the form of analytical expressions showing the explicit influence of every parameter of the problem, together with conditions specifying the range of validity, and conditions guaranteeing that fusion or sublimation develop the necessary supply of lubricant. For substances like water and bismuth that contract upon melting, we extend the results to the interesting range of conditions dominated by the effect of pressure on the melting temperature.*

## Introduction

The purpose of this paper is to study the hydrodynamic lubrication of solid-solid sliding bearings sustained by a fluid film produced by fusion or sublimation of the slider substance.

Lubrication by melting has been studied for a variety of applications, including ice-skating, special metal forming processes, and metal cutting applications [1-6]. A unique advantage of lubrication by melting or sublimation, is that the lubricant is produced in the region of contact exactly where it is needed between the two solid surfaces. Thus, if the physical and operating conditions are such that a sufficient supply of fluid is developed spontaneously by fusion or sublimation, then the mechanical and systemic difficulties typical of lubricant adduction in conventional lubrication systems can be overcome.

The driving forces for the spontaneous production of lubricant by fusion or sublimation are: imposed temperature differences between the two surfaces in contact; viscous dissipation in the fluid film between the two surfaces; and spontaneous temperature differences due to the Clausius-Clapeyron effect of pressure in the region of contact on the melting temperature, for those substances, like water and bismuth, for which an increased pressure reduces the melting temperature. In this paper, we study the relative importance of these three driving forces, and obtain expressions for the friction coefficient as a function of the operating conditions. Our analysis extends that in the literature, especially references 5 and 6, to the case of sublimation and to the range of conditions where the Clausius-Clapeyron effect cannot be neglected. In addition, we give explicit conditions defining the range of validity of the various resulting expressions.

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## Assumptions and Governing Equations

The object of our analysis (Fig. 1) is the fluid film maintained at steady state between the solid slider and the rigid track by fusion or sublimation of the slider substance.

In this section, we write the governing equations (mass, momentum, and energy balance) under a set of reasonable assumptions. The range of validity of the assumptions will be verified later for each of the approximate solutions obtained.

**Continuity (for the Solid Slider).** The solid slider moves toward the track with a bodily velocity  $V = w/\rho_s$ , where  $\rho_s$  is the density of the solid slider material and  $w$  the rate at which mass melts or sublimates at the solid-fluid interface. At steady state,  $V$  is a constant and so is  $w$ . In particular,

$$\frac{dw}{dx} = 0 \quad (1)$$

where  $x$  is the horizontal coordinate (Fig. 1).

**Continuity (for the Fluid Film).** We neglect side leakage effects, i.e., we assume that the slider is much wider than  $L$ . We assume that the flow is incompressible, i.e., that  $\Delta\rho/\rho \ll 1$ , even though the fluid density is not necessarily constant. We assume that the local slope of the solid-fluid inter-

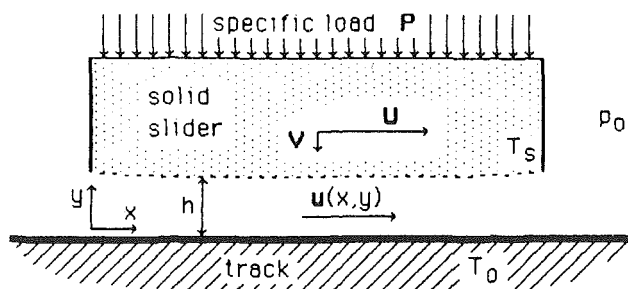


Fig. 1

face is small, i.e., that  $|dh(x)/dx| \ll 1$ , where  $h(x)$  is the fluid film thickness at  $x$ .

Under the above assumptions, the continuity equation for the fluid film at steady state becomes

$$\frac{d}{dx} \int_0^{h(x)} \rho(x,y)u(x,y)dy = w \quad (2)$$

where  $\rho(x,y)$  is the fluid density field and  $u(x,y)$  the horizontal velocity field.

**Momentum Balance.** We assume that the fluid film flow is locally inertia free, i.e., that the ratio of convective to viscous terms in the Navier-Stokes equations is negligible. This implies the condition

$$\frac{wh}{\mu} \ll 1 \quad (3)$$

The range of validity of this condition will be specified by equation (19).

Again, we assume that the flow is locally fully developed, i.e., that the local slope of the solid-fluid interface is small,  $|dh/dx| \ll 1$ , and that  $h \ll L$ . These conditions will determine the range of validity of our solutions and will be verified later.

The assumptions imply that  $\partial p/\partial y = 0$ , i.e., the fluid pressure is a function of  $x$  only, and that

$$u(x,y) = U \frac{y}{h} - \frac{p'h^2}{2\mu} \frac{y}{h} \left(1 - \frac{y}{h}\right) \quad (4)$$

where  $U$  is the horizontal slider velocity,  $y$  is the vertical coordinate defined in Fig. 1,  $h = h(x)$ ,  $p' = p'(x) = dp(x)/dx$ , and  $\mu$  is the fluid viscosity (assumed constant).

The friction coefficient is given by

$$\begin{aligned} f &= \frac{F}{P} = \frac{1}{PL} \int_0^L \mu \left( \frac{\partial u}{\partial y} \right)_{y=0} dx \\ &= \frac{\mu U}{Ph} - \frac{1}{2PL} \int_0^L p' h dx = \frac{\mu U}{Ph} \end{aligned} \quad (5)$$

where again we used the assumption  $|dh/dx| \ll 1$ .

The steady-state lubrication by the fluid film can be self-sustained only if the mean fluid speed at  $x = L$  is greater than  $U$ , i.e.,

$$\bar{u}_L = \frac{\int_0^{h(L)} \rho(L,y)u(L,y)dy}{\int_0^{h(L)} \rho(L,y)dy} > U \quad (6)$$

so that the fluid produced by fusion or sublimation provides a

sufficient supply of lubricant. Otherwise, steady-state lubrication can only be maintained if the track surface ahead of the slider is already covered by a fluid film of the slider substance (or other lubricant).

**Energy Balance.** We will assume that the energy flow is conduction dominated, i.e., that the ratio of convective to conductive terms in the energy equation is negligible. This implies the condition

$$\frac{wh}{\mu} \frac{\mu c}{k} \ll 1 \quad (7)$$

where  $\mu c/k$  is the Prandtl number of the fluid. The range of validity of this condition will be specified by equation (20).

Under the assumption that  $|dh/dx| \ll 1$ , the energy equation reduces to

$$0 = k \frac{\partial^2 T}{\partial y^2} + \mu \left( \frac{\partial u}{\partial y} \right)^2 \quad (8)$$

with boundary conditions

$$T(x,0) = T_0 \quad (9)$$

$$T(x,h(x)) = T_0 - \Delta T(x) \quad (10)$$

$$-k \frac{\partial T}{\partial y} \Big|_{y=h(x)} = wH \quad (11)$$

where  $T_0$  is the temperature imposed on the rigid track,  $k$  is the fluid thermal conductivity,  $H$  is the enthalpy of fusion or sublimation, and

$$\Delta T(x) = \Delta T_0 + \frac{T_s(\rho_0 - \rho_s)/\rho_s}{\rho_0 H} [p(x) - p_0] \quad (12)$$

$\Delta T_0 = T_0 - T_s$ ,  $T_s$  is the fusion or sublimation temperature,  $\rho_s$  is the solid-phase density of the slider substance, and  $\rho_0$  the fluid density at  $T_s$  and  $p_0$ . The second term in the right-hand side of equation (12) is a linear approximation of the Clausius-Clapeyron relation.

Equations (8) to (11), combined with equation (4) yield

$$wH = \frac{k\Delta T}{h} + \frac{\mu U^2}{2h} + \frac{p'Uh}{6} + \frac{p'^2 h^3}{24\mu} \quad (13)$$

where  $\Delta T = \Delta T(x)$ ,  $h = h(x)$ , and  $p' = p'(x) = dp(x)/dx$ .

**Equation of State.** We will consider two cases: (a) the fluid is a constant density liquid (fusion), and (b) the fluid is an ideal gas (sublimation).

**Liquid.** The density field

$$\rho(x,y) = \rho_0 \quad (14)$$

where  $\rho_0$  is the constant density of the fluid. Equations (2), (4), and (14) yield

## Nomenclature

$$A = \frac{kT_s(P/\rho_0 H)(\rho_0 - \rho_s)/\rho_s}{k\Delta T_0 + \mu U^2/2},$$

dimensionless parameter

$$F = \frac{1}{L} \int_0^L \mu \left( \frac{\partial u}{\partial y} \right)_{y=0} dx,$$

average friction force per unit of sliding surface area

$H = h_{sf}$ , enthalpy of fusion or sublimation

$L$  = length of solid slider

$$P = \frac{1}{L} \int_0^L (p - p_0) dx,$$

specific load, load per unit of sliding surface area

$R$  = gas constant for the slider substance

$T_0$  = rigid track temperature

$T_s$  = solid slider

surface temperature at  $x = L$ , melting or sublimation temperature at  $p_0$

$U$  = horizontal speed of slider

$V = w/\rho_s$ , vertical downward speed of slider

$f = F/P$ , effective friction coefficient

$h$  = fluid film thickness

$k$  = fluid conductivity

$p$  = pressure

$p_0$  = environmental pressure

$u$  = horizontal fluid speed

$\bar{u}_L$  = mean fluid speed at  $x = L$

$x$  = horizontal coordinate

$y$  = vertical coordinate

$w = \rho_s V$ , rate of solid mass melting or sublimating per unit of sliding surface area

$\Delta T_0 = T_0 - T_s$ , temperature difference across fluid film at  $x = L$

$\mu$  = fluid viscosity

$\rho$  = fluid density

$\rho_0$  = fluid density at  $p_0$  and  $T_s$

$\rho_s$  = density of solid slider material

**Table 1**

	Units	H <sub>2</sub> O	Bi	Na	CO <sub>2</sub>	
$T_s$	fusion temperature at $p_0$	[K]	273	544	371	--
$H$	enthalpy of fusion at $p_0$	[J/kg]	$3.34 \times 10^5$	$5.02 \times 10^4$	$1.94 \times 10^6$	--
$T_s$	sublimation temperature at $p_0$	[K]	--	--	--	193
$H$	enthalpy of sublimation at $p_0$	[J/kg]	--	--	--	$5.73 \times 10^6$
$\mu$	viscosity at $T_s$ and $p_0$	[kg/ms]	$1.79 \times 10^{-3}$	$1.75 \times 10^{-3}$	$7.10 \times 10^{-4}$	$9.25 \times 10^{-6}$
$k$	thermal conductivity at $T_s$ and $p_0$	[J/msK]	0.564	16.8	88	$9.09 \times 10^{-3}$
$\mu c/k$	Prandtl number at $T_s$ and $p_0$	[--]	13.4	0.015	0.011	0.817
$\rho_s$	density of solid phase	[kg/m <sup>3</sup> ]	917	9750	970	1560
$\rho_0$	density of fluid phase at $T_s$ and $p_0$	[kg/m <sup>3</sup> ]	1000	10069	929	2.75
$\rho_0 H$		[MPa]	334	506	1802	1.6

$$w = \rho_0 \frac{d}{dx} \left[ \frac{Uh}{2} - \frac{p'h^3}{12\mu} \right] \quad (15)$$

where  $h = h(x)$  and  $p' = p'(x)$ . Equations (1), (13), and (15) are the governing equations in the liquid film case.

**Gas.** The density field

$$\rho(x,y) = \frac{p(x)}{RT(x,y)} \quad (16)$$

will be approximated by  $\rho(x,y) = p(x)/RT_m(x)$ , assuming  $T(x,y) = T_m(x) = T_0 - 1/2\Delta T(x)$ . Equations (2), (4) and (16) yield

$$w = \frac{d}{dx} \frac{p}{RT_m} \left[ \frac{Uh}{2} - \frac{p'h^3}{12\mu} \right] \quad (17)$$

where  $h = h(x)$ ,  $p = p(x)$ ,  $p' = p'(x)$  and  $T_m = T_m(x)$ . Equations (1), (13), and (17) are the governing equations in the gas film case.

**Dimensionless Parameter  $P/\rho_0 H$ .** Throughout our analysis we will assume

$$\frac{P}{\rho_0 H} \ll 1 \quad (18)$$

Using data from Table 1, we see that, for H<sub>2</sub>O, Bi, Na, and CO<sub>2</sub> Condition 18 requires  $P \ll 334$ MPa, 506MPa, 1802MPa, and 1.6MPa, respectively, and may be verified under a wide range of values of the specific load  $P$ .

Under Condition 18, Conditions 3 and 7 become

$$\frac{wh}{\mu} = \frac{k\Delta T_0 + \mu U^2/2}{\mu H} \ll 1 \quad (19)$$

and

$$\frac{wh}{\mu} \frac{\mu c}{k} = \frac{k\Delta T_0 + \mu U^2/2}{\mu H} \frac{\mu c}{k} \ll 1 \quad (20)$$

and may be verified under a wide range of values of the track temperature  $T_0$  and the slider speed  $U$ .

Based on Condition 18, we will assume that the fluid film thickness  $h$  is approximately constant, i.e., we will assume  $h(x) = h_0$  and find expressions for  $h_0$  in terms of operating conditions. A better approximation for  $h(x)$  is obtained (for the liquid case) by combining equations (1), (13), and (15) to yield a differential equation for  $h(x)$ . This is done for example, in references [5 and 6]. However, we can show that the next approximation for  $h(x)$  is given by

$$h(x) = h_0 \left[ 1 + 12 \frac{P}{\rho_0 H} \left( \frac{x}{L} - \frac{1}{2} - \delta \right)^2 \right]$$

where  $h_0$  is the value we will find, and  $\delta$  is a small number ( $\ll 1$ ) depending on operating conditions. Thus, we conclude that the approximation  $h(x) = h_0$  is consistent with the assumption  $P/\rho_0 H \ll 1$  and is verified under a wide range of conditions.

**Dimensionless Parameter  $A$ .** The effect of pressure on melting or sublimation temperature is controlled by the dimensionless ratio

$$A = \frac{kT_s \frac{\rho_0 - \rho_s}{\rho_s} \frac{P}{\rho_0 H}}{k\Delta T_0 + \mu U^2/2} \quad (21)$$

The value of  $A$  may be positive or negative depending on the value of the imposed temperature difference  $\Delta T_0$  and, in the case of fusion, on whether the slider substance expands or contracts upon melting.

**Liquid or Gaseous Self-Lubrication With  $|A| \ll 1$**

In this section, we find approximate solutions that are valid if the Clausius-Clapeyron effect of pressure on the phase-transition temperature can be neglected, as was assumed in the analyses in references [5 and 6]. This applies to conditions of slider speed  $U$ , track temperature  $T_0$ , and specific load  $P$  such that, for the particular slider substance, the dimensionless parameter  $A$  is negligible.

Eliminating  $w$  from equations (13) and (15) under the assumption that  $P/\rho_0 H \ll 1$  and  $|A| \ll 1$ , yields for the liquid film case,

$$\frac{k\Delta T_0}{h} + \frac{\mu U^2}{2h} = \rho_0 H \frac{d}{dx} \left( \frac{Uh}{2} - \frac{p'h^3}{12\mu} \right) \quad (22)$$

Similarly, eliminating  $w$  from equations (13) and (17) under the same assumptions, yields, for the gaseous film case,

$$\frac{k\Delta T_0}{h} + \frac{\mu U^2}{2h} = \rho_m H \frac{d}{dx} \frac{p}{\rho_0} \left( \frac{Uh}{2} - \frac{p'h^3}{12\mu} \right) \quad (23)$$

where  $\rho_m = 2\rho_0/R(T_0 + T_s)$  is the mean density of the gaseous film.

Though equations (22) and (23) for the liquid and the gaseous case are different, we may verify that they both admit a solution with  $h(x) = h_0$  and

$$p(x) - p_0 = 6P \frac{x}{L} \left( 1 - \frac{x}{L} \right) \quad (24)$$

that is consistent with Condition 18, with  $|A| \ll 1$ , and with equations (1) and (13). The relations between the specific load  $P$  and the film thickness  $h_0$  are, however, different in the two cases.

**Liquid ( $|A| \ll 1$ ).** The solution is

$$h_0 = \left[ \frac{\mu L^2 (k\Delta T_0 + \mu U^2/2)}{\rho_0 H P} \right]^{1/4} \quad (25)$$

$$w = \left[ \frac{\rho_0 P (k\Delta T_0 + \mu U^2/2)^3}{\mu H^3 L^2} \right]^{1/4} \quad (26)$$

$$f = \left[ \frac{\mu^3 \rho_0 H U^4}{L^2 P^3 (k \Delta T_0 + \mu U^2 / 2)} \right]^{1/4} \quad (27)$$

$$\frac{\dot{u}_L}{U} = \frac{1}{2} + \frac{1}{2} \left[ \frac{k \Delta T_0 + \mu U^2 / 2}{\mu U^2} \frac{P}{\rho_0 H} \right]^{1/2} \quad (28)$$

The dependence of the friction coefficient  $f$  and the film thickness  $h_0$  on the slider speed  $U$ , slider length  $L$  and specific load  $P$  are in perfect agreement with the results plotted in reference [5] for the case  $\Delta T_0 = 0$  and in reference [6] for  $\Delta T_0 \neq 0$ .

The self-sustaining lubrication Condition 6 requires

$$\frac{k \Delta T_0 + \mu U^2 / 2}{\mu U^2} \frac{P}{\rho_0 H} > 1 \quad (29)$$

In view of the conditions  $P/\rho_0 H \ll 1$  and  $|A| \ll 1$ , the above solution describes a self-sustained steady lubrication by fusion provided that

$$\frac{\mu U^2}{k \Delta T_0} < \frac{P}{\rho_0 H} \ll \min \left( \frac{\Delta T_0}{T_s} \frac{\rho_s}{|\rho_0 - \rho_s|}, 1 \right) \quad (30)$$

For example, for H<sub>2</sub>O, Bi, and Na, using data from Table 1, the ranges of validity of the solution are, respectively,

$$\text{H}_2\text{O: } 1.06 \frac{U^2}{\Delta T_0} < P < \min(13.5 \Delta T_0, 334)$$

$$f = 4.3 \times 10^{-5} U P^{-3/4} L^{-1/2} \Delta T_0^{-1/4}$$

$$\text{Bi: } 0.0527 \frac{U^2}{\Delta T_0} < P < \min(28.4 \Delta T_0, 506)$$

$$f = 2.0 \times 10^{-5} U P^{-3/4} L^{-1/2} \Delta T_0^{-1/4}$$

$$\text{Na: } 0.0145 \frac{U^2}{\Delta T_0} < P < \min(115 \Delta T_0, 1798)$$

$$f = 9.2 \times 10^{-6} U P^{-3/4} L^{-1/2} \Delta T_0^{-1/4}$$

with  $U$  in m/s,  $\Delta T_0$  in K,  $P$  in MPa, and  $L$  in m.

**Gas ( $|A| \ll 1$ ).** The solution is

$$h_0 = \left[ \frac{\mu L^2 (k \Delta T_0 + \mu U^2 / 2)}{\rho_m H P (1 + 3P/2p_0)} \right]^{1/4} \quad (31)$$

$$w = \left[ \frac{\rho_m P (k \Delta T_0 + \mu U^2 / 2)^3 (1 + 3P/2p_0)}{\mu H^3 L^2} \right]^{1/4} \quad (32)$$

$$f = \left[ \frac{\mu^3 \rho_m H (1 + 3P/2p_0) U^4}{L^2 P^3 (k \Delta T_0 + \mu U^2 / 2)} \right]^{1/4} \quad (33)$$

$$\frac{\dot{u}_L}{U} = \frac{1}{2} + \frac{1}{2} \left[ \frac{k \Delta T_0 + \mu U^2 / 2}{\mu U^2} \frac{P}{\rho_m H (1 + 3P/2p_0)} \right]^{1/2} \quad (34)$$

The self-sustaining lubrication Condition 6 requires

$$\frac{k \Delta T_0 + \mu U^2 / 2}{\mu U^2} \frac{P}{\rho_m H (1 + 3P/2p_0)} > 1 \quad (35)$$

In view of the conditions  $P/\rho_0 H \ll 1$  and  $|A| \ll 1$  the above solution describes a self-sustained steady lubrication by sublimation provided that

$$\frac{\mu U^2}{k \Delta T_0} \left( 1 + \frac{3P}{2p_0} \right) < \frac{P}{\rho_m H} \ll \min \left( \frac{\Delta T_0}{T_s} \frac{\rho_s}{|\rho_0 - \rho_s|}, 1 \right) \quad (36)$$

For example, for CO<sub>2</sub> the range of validity of the solution is

$$\text{CO}_2: 0.0016 \frac{U^2}{\Delta T_0} (1 + 15P) < P < \min(0.008 \Delta T_0, 1.6)$$

$$f = 6 \times 10^{-7} U P^{-3/4} (1 + 15P)^{1/4} L^{-1/2} \Delta T_0^{-1/4}$$

with  $U$  in m/s,  $\Delta T_0$  in K,  $P$  in MPa, and  $L$  in m (we used  $\rho_m = \rho_0$  for an order of magnitude estimate).

## Liquid Self-Lubrication With Arbitrary Positive $A$

In this section, we study conditions in which the Clausius-Clapeyron effect of pressure on the melting temperature at the solid-liquid interface cannot be neglected. We restrict our analysis to the most interesting case of slider substances, like water and bismuth, which expand upon solidification, i.e., such that  $\rho_0 > \rho_s$ .

Eliminating  $w$  from equations (13) and (15) under the assumption that  $P/\rho_0 H \ll 1$ , and using equation (12) for  $\Delta T$ , yields

$$\begin{aligned} \frac{k T_s (\rho_0 - \rho_s) / \rho_s}{h} \frac{p - p_0}{\rho_0 H} + \frac{k \Delta T_0}{h} + \frac{\mu U^2}{2h} \\ = \rho_0 H \frac{d}{dx} \left( \frac{Uh}{2} - \frac{\rho' h^3}{12\mu} \right) \end{aligned} \quad (37)$$

Again, this equation admits a solution with  $h(x) = h_0$  that is consistent with Condition 18, and equations (1) and (13). To find it, we rewrite equation (37) as follows

$$\frac{d^2 p}{dx^2} + \frac{4B^2}{L^2} (p - p_0) + \frac{4B^2}{L^2} \frac{P}{A} = 0 \quad (38)$$

where  $A$  is defined by equation (21),  $P$  is the specific load, and

$$B^2 = \frac{3\mu L^2 k T_s (\rho_0 - \rho_s) / \rho_s}{(\rho_0 H)^2 h_0^4} \quad (39)$$

The solution of equation (38) is

$$p(x) - p_0 = \frac{P}{A} \left[ (tgB) \sin 2B \frac{x}{L} + \cos 2B \frac{x}{L} - 1 \right] \quad (40)$$

and, therefore,

$$P = \frac{1}{L} \int_0^L (p - p_0) dx = \frac{P}{A} \left[ \frac{1}{B} tgB - 1 \right] \quad (41)$$

which yields the equation

$$\frac{1}{B} tgB = 1 + A \quad (42)$$

For each given set of operating conditions, parameter  $A$  (equation (21)) is fixed, and the solution of equation (42) yields  $B = \bar{B}(A)$ . It is readily seen that  $\bar{B}(A) = (3A)^{1/2}$  for  $A \ll 1$  and the pressure distribution equation (40) reduces to equation (24) for  $A \ll 1$ . On the other hand,  $\bar{B}(A) = \pi/2$  (independent of  $A$ ) for  $A \gg 1$ .

Once  $\bar{B}(A)$  is determined, equation (39) yields the film thickness

$$h_0 = \left[ \frac{3\mu L^2 k T_s (\rho_0 - \rho_s) / \rho_s}{\bar{B}^2(A) (\rho_0 H)^2} \right]^{1/4} \quad (44a)$$

$$= \left[ \frac{3A\mu L^2 (k \Delta T_0 + \mu U^2 / 2)}{\bar{B}^2(A) \rho_0 H P} \right]^{1/4} \quad (43b)$$

Clearly, for  $A \ll 1$ , equation (43b) reduces to equation (25). But for increasing values of  $A$ ,  $h_0$  tends to a value independent of  $A$  and, therefore, independent of  $P$ ,  $\Delta T_0$  and  $U$ . This value depends only on the physical parameters of the substance and the slider length  $L$ . For example, for H<sub>2</sub>O and Bi, using data from Table 1, we find

$$\text{H}_2\text{O: } h_0(A \gg 1) = 2.28 \times 10^{-5} L^{1/2}$$

$$\text{Bi: } h_0(A \gg 1) = 3.97 \times 10^{-5} L^{1/2}$$

with  $h_0$  and  $L$  in m.

Substituting  $h_0$  from equation (43) into equation (13) (averaged over  $L$ ), yields

$$w = (1 + A) \left[ \frac{\bar{B}^2(A) \rho_0 P (k \Delta T_0 + \mu U^2 / 2)^3}{3A\mu H^3 L^2} \right]^{1/4} \quad (44a)$$

$$= \left(1 + \frac{1}{A}\right) \left[ \frac{\bar{B}^2(A) \rho_0 P (kT_s (\rho_0 - \rho_s) / \rho_s)^3 (P / \rho_0 H)^3}{3\mu H^3 L^2} \right]^{1/4} \quad (44b)$$

Equation (5) yields

$$f = \left[ \frac{\bar{B}^2(A) \mu^3 \rho_0^2 H^2 U^4}{3L^2 P^4 kT_s (\rho_0 - \rho_s) / \rho_s} \right]^{1/4} \quad (45a)$$

$$= \left[ \frac{\bar{B}^2(A) \mu^3 \rho_0 H U^4}{3AL^2 P^3 (k\Delta T_0 + \mu U^2 / 2)} \right]^{1/4} \quad (45b)$$

Equation (4), together with equations (40) and (42), yields,

$$\frac{\bar{u}_L}{U} = \frac{1}{2} + \frac{1}{2} (1 + A) \left[ \frac{\bar{B}^2(A) (k\Delta T_0 + \mu U^2 / 2) P}{3A\mu U^2 \rho_0 H} \right]^{1/2} \quad (46a)$$

$$= \frac{1}{2} + \frac{1}{2} \left(1 + \frac{1}{A}\right) \frac{P}{\rho_0 H} \left[ \frac{\bar{B}^2(A) kT_s (\rho_0 - \rho_s) / \rho_s}{3\mu U^2} \right]^{1/2} \quad (46b)$$

Lubrication is self-sustained if  $\bar{u}_L / U > 1$ , i.e., fusion develops a sufficient supply of fluid to maintain a positive flow rate at the leading edge of the slider.

The self-sustaining lubrication Condition 6, for  $A \ll 1$  yields Condition 29 and leads to the range of validity of our approximate solution for  $A \ll 1$  (equations (24) to (28)) given by Condition 30.

In the limit of  $A \gg 1$ , instead, Condition 6 requires

$$\left[ \frac{12\mu U^2}{\pi^2 kT_s (\rho_0 - \rho_s) / \rho_s} \right]^{1/2} < (P / \rho_0 H) < 1 \quad (47)$$

Thus, for H<sub>2</sub>O and Bi, the approximate solution given by equations (43)–(46) with  $A \gg 1$  and  $\bar{B}(A) = \pi/2$  describes a self-sustained steady lubrication by fusion provided that the following conditions are satisfied, where we also give expressions for the friction coefficient

$$\begin{aligned} \text{H}_2\text{O}: \quad & 4.2U < P < < 334 \\ & 13.5\Delta T_0 + 0.021U^2 < < P \\ & f = 7.8 \times 10^{-5} UP^{-1} L^{-1/2} \end{aligned} \quad (A \gg 1)$$

$$\begin{aligned} \text{Bi}: \quad & 1.35U < P < < 506 \\ & 28.4\Delta T_0 + 0.0015U^2 < < P \\ & f = 4.4 \times 10^{-5} UP^{-1} L^{-1/2} \end{aligned} \quad (A \gg 1)$$

with  $U$  in m/s,  $P$  in MPa,  $\Delta T_0$  in K, and  $L$  in m.

## Conclusions

The friction coefficients obtained in solid slider bearings lubricated by their own melting or sublimation can be extremely low. They range from  $10^{-2}$  to  $10^{-5}$ , depending on operating conditions. In practical applications, the adoption of such bearings depends on the break-even point between the cost and power dissipation of a conventional lubrication system, and the cost and power consumption of a fusion or sublimation bearing. Indeed, the power dissipation is reduced by a very low friction coefficient, but the slider material that is continuously consumed must be regenerated, and this requires in general a cost and a substantial power consumption. We have not addressed the problem of determining such a break-even point.

We have studied the thermodynamics and fluid mechanics of the self-development of the hydrodynamic lubrication by fusion and sublimation of the solid slider bearing under a wide range of operating conditions. We obtained expressions for the friction coefficient, and conditions for the range of operating conditions that guarantee a self-sustained steady-state lubrication.

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