A GENERAL NONLINEAR EVOLUTION EQUATION

FOR IRREVERSIBLE CONSERVATIVE APPROACH TO STABLE EQUILIBRIUM

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INTRODUCTION

The problem of understanding entropy and irreversibility has been tackled by thousands of physicists during the past century. Schools of thought have formed and flourished around different perspectives of the problem. But a definitive solution has yet to be found.[1]

We address a mathematical problem very relevant to the question of nonequilibrium and irreversibility, namely, that of "designing" a general evolution equation capable of describing irreversible but conservative relaxation towards equilibrium. Our objective is to present an interesting mathematical solution to this "design" problem, namely, a new nonlinear evolution equation that satisfies a set of very stringent relevant requirements.

In this lecture, we do not claim any physical meaning for the proposed nonlinear evolution equation. Indeed, we define three essentially different frameworks within which the new equation could be adopted, with entirely different interpretations. We purposely devoid this presentation of our school's quite unorthodox perspective on the physical meaning of entropy and irreversibility, because we feel that the proposed nonlinear equation constitutes an important advance in itself, independently of the physical context for which it was designed and developed.[2-4]

The lecture is organized as follows. First, we define three familiar mathematical frameworks within which the subsequent results may be interpreted. Then, we list the "design specifications" that we intend to impose on the desired evolution equation. We review

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some useful well-known mathematics involving Gram determinants and, finally, present our nonlinear evolution equation which meets the stringent design specifications.

Our views and hypotheses on entropy, nonequilibrium and irreversibility will be discussed in our second lecture in this volume.

FRAMEWORK A: QUANTUM STATISTICAL MECHANICS AND QUANTUM THERMODYNAMICS

Let $\mathcal{H}$ be a Hilbert space ($\dim \mathcal{H} \leq \infty^5$), and $\mathcal{L}$ the set of all linear operators A, B, ... on $\mathcal{H}$, equipped with the real inner product $(\cdot|\cdot)$ defined by

$$(A|B) = \frac{1}{2}\mathrm{Tr}(A^\dagger B + B^\dagger A) \tag{1a}$$

where $A^\dagger$ denotes the adjoint of operator A and Tr the trace functional. We denote by $\mathcal{P}$ the subset of all self-adjoint, nonnegative-definite, unit-trace operators ρ in $\mathcal{L}$, i.e.,

$$\mathcal{P} = \{\rho \text{ in } \mathcal{L} \,|\, \rho^\dagger = \rho,\ \rho \geq 0,\ \mathrm{Tr}\rho = 1\} \tag{2a}$$

We will then consider a set

$$\{H, N_1, \ldots, N_r\} \tag{3a}$$

of self-adjoint operators in $\mathcal{L}$, where each N_i commutes with H, i.e., is such that $HN_i = N_iH$, for $i = 1, \ldots, r$.

In Quantum Statistical Mechanics,[1] ρ is the von Neumann statistical or density operator which represents the index of statistics from a generally heterogeneous ensemble of identical systems (with associated Hilbert space $\mathcal{H}$) distributed over a range of possible quantum mechanical states.

In Quantum Thermodynamics,[3] ρ is the state operator which represents the individual quantum state of a strictly isolated system with associated Hilbert space $\mathcal{H}$, in the same sense as in Quantum Mechanics a vector ψ in $\mathcal{H}$ represents the individual quantum state.

In both theories, H is the Hamiltonian operator, and operator N_i, for $i = 1, \ldots, r$, is the number operator for particles of type i in the system (if the system has a fixed number n_i of particles of type i, then $N_i = n_iI$, where I is the identity operator on $\mathcal{H}$).

FRAMEWORK B: CLASSICAL STATISTICAL MECHANICS

Let Ω be a phase space, and $\mathcal{L}$ the set of real, square-integrable functions A, B, ... on Ω, equipped with the inner product $(\cdot|\cdot)$ defined by

$$(A|B) = \mathrm{Tr}AB = \int_\Omega AB d\Omega \tag{1b}$$

where Tr in this framework denotes $\int_\Omega d\Omega$. We denote by $\mathcal{P}$ the subset of all nonnegative-definite, normalized functions ρ in $\mathcal{L}$, i.e.,

$$\mathcal{P} = \{\rho \text{ in } \mathcal{L} \,|\, \rho \geq 0,\ \int_\Omega \rho d\Omega = 1\} \tag{2b}$$

We will then consider a set

$$\{H, N_1, \ldots, N_r\} \tag{3b}$$

of functions in $\mathcal{L}$.

In Classical Statistical Mechanics, ρ is the Gibbs density-of-phase function which represents the index of statistics from a generally heterogeneous ensemble of identical systems (with associated phase space Ω) distributed over a range of possible classical mechanical states. H is the Hamiltonian function, and N_i the number function for particles of type i.

FRAMEWORK C: INFORMATION THEORY

Let $\mathcal{L}$ be the set of all $n \times n$ real, diagonal matrixes $A = \mathrm{diag}(a_j)$, $B = \mathrm{diag}(b_j)$, ... $(n \leq \infty)$, equipped with the inner product $(\cdot|\cdot)$ defined by

$$(A|B) = \mathrm{Tr}AB = \sum_{j=1}^{n} a_j b_j \tag{1c}$$

We denote by $\mathcal{P}$ the subset of all nonnegative-definite, unit-trace matrixes ρ in $\mathcal{L}$, i.e.,

$$\mathcal{P} = \{\rho = \mathrm{diag}(p_j) \,|\, p_j \geq 0,\ \mathrm{Tr}\rho = \sum_{j=1}^{n} p_j = 1\} \tag{2c}$$

We will then consider a set

$$\{H, N_1, \ldots, N_r\} \tag{3c}$$

of diagonal matrixes $H = \mathrm{diag}(e_j)$, $N_1 = \mathrm{diag}(n_{1j})$, ..., $N_r = \mathrm{diag}(n_{rj})$ in $\mathcal{L}$.

In Information Theory,[6] $\rho = \mathrm{diag}(p_j)$ represents the probability assignment to a set of n events, p_j being the probability of occurrence of the j-th event. H, N_1, ..., N_r are characteristic features of the events in the set, taking on the values e_j, n_{1j}, ..., n_{rj}, respectively, for the j-th event.

MEAN VALUE FUNCTIONALS AND S-FUNCTIONAL

From here on, our notation allows us to treat at once the three frameworks just defined. For reasons to become apparent below, we call the elements H, N_1, ..., N_r in Equations 3 the generators of the motion. We will assume that such sets always contain at least element H, that we call the Hamiltonian generator of the motion.

For each generator of the motion, we then define a mean value functional on $\mathcal{P}$ as follows

$$\begin{aligned} m(\rho;H) &= \mathrm{Tr}\rho H = (\sqrt{\rho}\,|\,\sqrt{\rho}H) \\ m(\rho;N_i) &= \mathrm{Tr}\rho N_i = (\sqrt{\rho}\,|\,\sqrt{\rho}N_i) \end{aligned} \tag{4}$$

Moreover, we define the S-functional on $\mathcal{P}$ as

$$S(\rho) = -k\mathrm{Tr}\rho\ln\rho = -k(\sqrt{\rho}\,|\,\sqrt{\rho}\ln\rho) \tag{5}$$

Depending on the context, the S-functional represents the thermodynamic entropy, the statistical uncertainty as to the actual state of a system, or the information carried by the occurrence of one of the possible events.

For each given set of values $\langle H\rangle$, $\langle N_1\rangle$, ..., $\langle N_r\rangle$, in the range of the mean value functionals (Equations 4) corresponding to the generators of the motion, we consider the subset of all elements ρ in $\mathcal{P}$ that share the given mean values, i.e.,

$$\mathcal{P}(\langle H\rangle,\langle N_1\rangle,\ldots,\langle N_r\rangle) =$$

$$\{\rho \text{ in } \mathcal{P}\,|\,m(\rho;H) = \langle H\rangle,\ m(\rho;N_i) = \langle N_i\rangle \text{ for } i = 1,\ldots,r\} \tag{6}$$

On each such subset, i.e., for fixed mean values $\langle H\rangle$, $\langle N_1\rangle$, ..., $\langle N_r\rangle$ of the generators of the motion, the S-functional (Equation 5) achieves a unique maximum at the point

$$\rho = e^{-\alpha}\exp(-\beta H + \sum_{i=1}^{r}\nu_i N_i) \tag{7}$$

where

$$\alpha = \ln \mathrm{Tr} \exp(-\beta H + \sum_{i=1}^{r} \nu_i N_i) \tag{8}$$

$$\beta = \beta(\langle H\rangle, \langle N_1\rangle, \ldots, \langle N_r\rangle) \tag{9}$$

$$\nu_i = \nu_i(\langle H\rangle, \langle N_1\rangle, \ldots, \langle N_r\rangle) \tag{10}$$

It is noteworthy that the maximum-S points satisfy the condition

$$\sqrt{\rho}\ln\rho = -\alpha\sqrt{\rho} - \beta\sqrt{\rho}H + \sum_{i=1}^{r} \nu_i \sqrt{\rho}N_i \tag{11}$$

where α, β and ν_i, for $i = 1, \ldots, r$, are real numbers. In words, the maximum-S element ρ is such that $\sqrt{\rho}\ln\rho$ lies on the linear manifold generated by elements $\sqrt{\rho}$, $\sqrt{\rho}H$, $\sqrt{\rho}N_1$, ..., $\sqrt{\rho}N_r$. This observation will prove useful in what follows. In particular, it is noteworthy that Condition 11 is satisfied not only by the maximum-S elements given by Equation 7, but also by the elements given by

$$\rho = e^{-a} B \exp(-bH + \sum_{i=1}^{r} c_i N_i) \tag{12}$$

where

$$a = \ln \mathrm{Tr}\, B \exp(-bH + \sum_{i=1}^{r} c_i N_i) \tag{13}$$

$$b = b(B;\langle H\rangle, \langle N_1\rangle, \ldots, \langle N_r\rangle) \tag{14}$$

$$c_i = c_i(B;\langle H\rangle, \langle N_1\rangle, \ldots, \langle N_r\rangle) \tag{15}$$

and B is any idempotent element in $\mathcal{L}$ (i.e., $B^2 = B$) commuting with ρ (this last condition is trivally satisfied within Frameworks b and c). An element ρ satisfying Equation 12 maximizes S on $\mathcal{P}(\langle H\rangle, \langle N_1\rangle, \ldots, \langle N_r\rangle)$ and satisfies Equation 7 only if $B = I$ (I = identity operator on $\mathcal{H}$ in Framework a; I = constant function equal to 1 on the whole Ω in Framework b; I = diag(1) in Framework c).

DESIGN SPECIFICATIONS

Our "architectural" problem is to design a function $F(\cdot)$ such that every solution $\rho(t)$ of the autonomous differential equation

$$\frac{d}{dt}\rho(t) = F(\rho(t)) \tag{16}$$

with $\rho(0)$ in $\mathcal{D}(F) \subset \mathcal{P}$ [7] satisfies the following conditions for all $t \geq 0$:

(i) $\rho(t)$ lies entirely in $\mathcal{D}(F)$;

(ii) $m(\rho(t);H) = m(\rho(0);H)$, and $m(\rho(t);N_i) = m(\rho(0);N_i)$ for $i = 1,\ldots,r$;

(iii) $S(\rho(t+u)) \geq S(\rho(t))$ for all $u > 0$;

(iv) among all the elements ρ in $\mathcal{P}$ with given mean values $\langle H\rangle$, $\langle N_1\rangle$, ..., $\langle N_r\rangle$ of the generators of the motion, the unique maximum-S element $\rho = e^{-\alpha}\exp(-\beta H + \sum_{i=1}^{r}\nu_i N_i)$ (Equations 7 to 10) is the only equilibrium solution that is stable according to Liapunoff.[8]

It is noteworthy that requirement (iv) is most restrictive. For example, within Framework a, it rules out the von Neumann evolution equation ($F(\rho) = -i(H\rho-\rho H)/\hbar$) because all the equilibrium solutions (ρ such that $H\rho = \rho H$) are stable according to Liapunoff and, in general, there are many such solutions for each given set of mean values of the generators of the motion.

SOME NECESSARY MATHEMATICAL BACKGROUND

Given a subset of elements A, B, ..., Z in $\mathcal{L}$, we denote by $M(A,B,\ldots,Z)$ the Gram matrix

$$M(A,B,\ldots,Z) = \begin{bmatrix} (A|A) & (A|B) & \ldots & (A|Z) \\ (B|A) & (B|B) & \ldots & (B|Z) \\ \vdots & \vdots & \ddots & \vdots \\ (Z|A) & (Z|B) & \ldots & (Z|Z) \end{bmatrix} \tag{17}$$

where $(\cdot|\cdot)$ is the real symmetric inner product defined on $\mathcal{L}$. We denote by G(A,B,...,Z) the Gram determinant of A,B,...,Z with respect to the inner product $(\cdot|\cdot)$, i.e., G(A,B,...,Z) = det[M(A,B,...,Z)]. Matrix M(A,B,...,Z) is nonnegative definite and, therefore, its determinant G(A,B,...,Z) is also nonnegative. A necessary and sufficient condition for elements A,B,...,Z to be linearly independent is that their Gram determinant G(A,B,...,Z) be nonzero and, hence, strictly positive.

Given a subset of elements A, B, ..., Z in $\mathcal{L}$, we denote by L(A,B,...,Z) the linear manifold spanned by all linear combinations with real coefficients of the elements A, B, ..., Z. With respect to the inner product $(\cdot|\cdot)$ defined on $\mathcal{L}$, we denote the projection of a given element V in $\mathcal{L}$ onto a linear manifold L by the symbol $(V)_L$. $(V)_L$ is the unique element in L such that $((V)_L|X) = (V|X)$ for all X in L.

The theory of Gram determinants, very seldom used in the physics literature, offers a useful explicit way of writing the projection $(V)_L$ of V onto a given linear manifold L.

Let a linear manifold L be given, e.g., by specifying it as the set of all real linear combinations of given elements A, B, ..., Z in $\mathcal{L}$, not necessarily linearly independent, i.e., L = L(A,B,...,Z). Let us select any subset of linearly independent elements E_1, E_2, ..., E_m spanning L, i.e., such that $G(E_1,E_2,...,E_m) > 0$ and $L(E_1,E_2,...,E_m) = L$. By the definition of $(V)_L$, $((V)_L|E_j) = (V|E_j)$ for every j = 1, 2, ..., m, and $(V)_L = \sum_{i=1}^{m} v_i E_i$, where v_i are real scalars. Thus,

$$\sum_{i=1}^{m} v_i(E_i|E_j) = (V|E_j) \quad \text{for } j = 1, 2, \ldots, m \tag{18}$$

Because $(E_i|E_j) = [M(E_1,E_2,...,E_m)]_{ij}$ and the elements E_1, E_2, ..., E_m are linearly independent, Equations 18 are linearly independent and can be solved for the v_i's to yield

$$v_i = \sum_{j=1}^{m} (V|E_j)[M(E_1,E_2,...,E_m)^{-1}]_{ji} \qquad \text{for } i=1,2,...,m \tag{19}$$

and, therefore,

$$(V)_L = \sum_{i=1}^{m}\sum_{j=1}^{m} (V|E_j)[M(E_1,E_2,...,E_m)^{-1}]_{ji}E_i \tag{20}$$

An alternative, completely equivalent, but more elegant expression for $(V)_L$ is given by

$$(V)_L = -\frac{1}{G(E_1,E_2,\ldots,E_m)} \det \begin{bmatrix} 0 & E_1 & E_2 & \cdots & E_m \\ (E_1|V) & (E_1|E_1) & (E_1|E_2) & \cdots & (E_1|E_m) \\ (E_2|V) & (E_2|E_1) & (E_2|E_2) & \cdots & (E_2|E_m) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (E_m|V) & (E_m|E_1) & (E_m|E_2) & \cdots & (E_m|E_m) \end{bmatrix} \tag{21}$$

We next consider an important example that we shall use directly in what follows. For a given element ρ in the set $\mathcal{S}$, we will need to consider the projection of $\sqrt{\rho}\ln\rho$ onto the linear manifold $L(\sqrt{\rho},\sqrt{\rho}H,\sqrt{\rho}N_1,\ldots,\sqrt{\rho}N_r)$ where $\sqrt{\rho}$, $\sqrt{\rho}H$, $\sqrt{\rho}N_1$, ..., $\sqrt{\rho}N_r$ are not necessarily linearly independent. Using Equation 21 and Definitions 1 of the inner product $(\cdot|\cdot)$, we find

$$(\sqrt{\rho}\ln\rho)_{L(\sqrt{\rho}R_0,\sqrt{\rho}R_1,\ldots,\sqrt{\rho}R_z)} = -\frac{1}{G(\sqrt{\rho}R_0,\sqrt{\rho}R_1,\ldots,\sqrt{\rho}R_z)} \cdot$$

$$\cdot\det \begin{bmatrix} 0 & \sqrt{\rho}R_0 & \sqrt{\rho}R_1 & \cdots & \sqrt{\rho}R_z \\ \mathrm{Tr}\rho R_0\ln\rho & \mathrm{Tr}\rho R_0^2 & \frac{1}{2}\mathrm{Tr}\rho\{R_0,R_1\} & \cdots & \frac{1}{2}\mathrm{Tr}\rho\{R_0,R_z\} \\ \mathrm{Tr}\rho R_1\ln\rho & \frac{1}{2}\mathrm{Tr}\rho\{R_1,R_0\} & \mathrm{Tr}\rho R_1 & \cdots & \frac{1}{2}\mathrm{Tr}\rho\{R_1,R_z\} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{Tr}\rho R_z\ln\rho & \frac{1}{2}\mathrm{Tr}\rho\{R_z,R_0\} & \frac{1}{2}\mathrm{Tr}\rho\{R_z,R_1\} & \cdots & \mathrm{Tr}\rho R_z^2 \end{bmatrix} \tag{22}$$

where $\{A,B\} = AB + BA$, and the subset of elements R_0, R_1, ..., R_z in $\mathcal{L}$ is such that $L(\sqrt{\rho}R_0,\sqrt{\rho}R_1,\ldots,\sqrt{\rho}R_z) = L(\sqrt{\rho},\sqrt{\rho}H,\sqrt{\rho}N_1,\ldots,\sqrt{\rho}N_r)$ and $G(\sqrt{\rho}R_0,\sqrt{\rho}R_1,\ldots,\sqrt{\rho}R_z) > 0$.

With this background, we can present in a quite compact form an evolution equation meeting our very restrictive design specifications.

NEW NONLINEAR EVOLUTION EQUATION

The author designed and proposed the following evolution equation:[2,3]

$$\frac{d}{dt}\,\rho(t) = F(\rho(t)) \tag{23a}$$

$$F(\rho) = -\frac{i}{\hbar}[H,\rho] - \frac{1}{\tau}\frac{1}{2}(\sqrt{\rho}D(\rho) + D^{\dagger}(\rho)\sqrt{\rho}) \tag{23b}$$

$$D(\rho) = \sqrt{\rho}\ln\rho - (\sqrt{\rho}\ln\rho)_{L(\sqrt{\rho},\sqrt{\rho}H,\sqrt{\rho}N_1,\ldots,\sqrt{\rho}N_r)} \tag{23c}$$

where $[H,\rho] = H\rho - \rho H$ (= 0 within Frameworks b and c), $\hbar$ is the reduced Planck constant (playing a role only within Framework a), τ is a characteristic time constant, H, N_1, ..., N_r are fixed generators of the motion and, for each ρ, $L(\sqrt{\rho},\sqrt{\rho}H,\sqrt{\rho}N_1,\ldots,\sqrt{\rho}N_r)$ is the linear manifold generated by elements $\sqrt{\rho}$, $\sqrt{\rho}H$, $\sqrt{\rho}N_1$, ..., $\sqrt{\rho}N_r$. Function $F(\rho)$ is clearly nonlinear in ρ.

It is noteworthy that along any solution $\rho(t)$, $D(\rho(t))$ is the "instantaneous distance" element between element $\sqrt{\rho}(t)\ln\rho(t)$ and the linear manifold spanned by $\sqrt{\rho}(t)$, $\sqrt{\rho}(t)H$, $\sqrt{\rho}(t)N_1$, ..., $\sqrt{\rho}(t)N_r$. At each instant in time along a solution, the inner product of $i[H,\rho]$ and $\sqrt{\rho}D(\rho) + D^{\dagger}(\rho)\sqrt{\rho}$ is equal to zero and, therefore, the unitary contribution to $d\rho/dt$ due to the first linear term in Equation 23b is orthogonal to the nonunitary contribution to $d\rho/dt$ due to the second nonlinear term in Equation 23b.

The original motivation for the design of this new evolution equation, together with the physical justification of the design specifications, is discussed in our second lecture of this series, and in References 2 to 4. Here we are mainly concerned with the mathematical properties of the new evolution equation and its solutions. The properties that we list below emerge from our analysis of Equation 23 in References 3 and 9. In Reference 3, we adopt Equation 23 as the general quantum thermodynamic equation of motion for a single strictly isolated constituent of matter. Much technical mathematical work remains to be done on our conjectures in Reference 3. In Reference 9, by specializing the analysis to a single strictly isolated two-level system, we prove rigorously all the properties of the equation, including the existence and uniqueness of solutions to the initial value problem both in forward and in backward time.

For all t in the range $(-\infty,+\infty)$, i.e., both in forward and backward time, if $\rho(0)$ is in $\mathcal{D}(F) \subset \mathcal{P}$ then:

(i) $\rho(t)$ remains in $\mathcal{D}(F)$, i.e., there are no forward nor backward escape times;

(ii) the mean value of any linear combination of I, H, N_1, ..., N_r is time-invariant, i.e., for every set of real scalars a, b, c_1, ..., c_r,

$$\mathrm{Tr}\ \rho(t)[aI + bH + \sum_{i=1}^{r} c_i N_i] = \text{const.} \tag{24}$$

(iii) $S(\rho(t+u)) \geq S(\rho(t))$ for all $u > 0$, because

$$\frac{d}{dt}S(\rho) = (D(\rho)|D(\rho)) = \frac{G(\sqrt{\rho}\ln\rho,\sqrt{\rho}R_0,\sqrt{\rho}R_1,\ldots,\sqrt{\rho}R_z)}{G(\sqrt{\rho}R_0,\sqrt{\rho}R_1,\ldots,\sqrt{\rho}R_z)} \geq 0 \tag{25}$$

Moreover, the strict equality applies, i.e., $dS(\rho)/dt = 0$, if and only if ρ satisfies the condition

$$\sqrt{\rho}\ln\rho = -a\sqrt{\rho} - b\sqrt{\rho}H + \sum_{i=1}^{r} c_i\sqrt{\rho}N_i \tag{26}$$

for some real scalars a, b, c_1, ..., c_r, in which case ρ is said to be nondissipative, and can be written according to Equation 12;

(iv) every nondissipative ρ is either an equilibrium solution of Equation 23 or it belongs to a limit cycle. An equilibrium solution is unstable if ρ can be written according to Equation 12 with $B \neq I$, whereas it is stable if ρ can be written according to Equation 7.

In Reference 10, by specializing the analysis to a single strictly isolated N level system, we prove that Equation 23 implies a generalization of Onsager's reciprocal relations valid for all nonequilibrium states, close and far from stable equilibrium. Onsager's coefficients emerge as well-defined nonlinear functionals of the quantum thermodynamic state operator.

ADDITIONAL DESIGN SPECIFICATIONS

A generalization of Equation 23, consistent with additional design specifications required for the description of composite systems is proposed in References 1 and 11. There, we conjecture that the nonlinear term in the new evolution equation implies an intrinsic mechanism of loss of correlations between component subsystems.

A SIMPLEST EXAMPLE

As an illustrative example, let us select Framework c, with $n = 2$, $\rho = \mathrm{diag}(x, 1-x)$, $0 \leq x \leq 1$, $H = \mathrm{diag}(1,1)$ and $r = 0$. We discuss only the mathematical aspects of this simplest example, not its possible information-theoretic applications. Equation 23 becomes

$$\dot{x} = \begin{cases} 0 & \text{if } x = 0 \text{ or } x = 1 \\ -\frac{1}{\tau}\, x(1-x) \ln\frac{x}{1-x} & \text{if } 0 < x < 1 \end{cases} \tag{27}$$

There are three equilibrium solutions, $x = 0$, $x = 1$, and $x = 1/2$. The nonequilibrium solutions are

$$x(t) = \frac{[x_o/(1-x_o)]^{\exp(-t/\tau)}}{1 + [x_o/(1-x_o)]^{\exp(-t/\tau)}} \tag{28a}$$

or, equivalently,

$$\ln\frac{x(t)}{1-x(t)} = e^{-t/\tau} \ln\frac{x(0)}{1-x(0)} \tag{28b}$$

We easily verify that $x(t)$ is either always greater or always smaller than $1/2$, for $-\infty < t < +\infty$. The solutions are defined both in forward and backward time, with no finite escape times. If $x_o > 1/2$, then $x(t) \to 1$ as $t \to -\infty$. If $x_o < 1/2$, then $x(t) \to 0$ as $t \to -\infty$. In either case, $x(t) \to 1/2$ as $t \to +\infty$. The only equilibrium solution which is stable is the maximum-S $x = 1/2$, whereas equilibrium solutions $x = 0$ and $x = 1$ are unstable.

CONCLUSIONS

We presented a nonlinear evolution equation new to physics. The new equation is applicable within the frameworks of Quantum Thermodynamics, Classical and Quantum Statistical Mechanics, and Information Theory. The nonlinear equation (and its generalization discussed in Reference 11) yields a unique deterministic description of irreversible, but conservative relaxation from nonequilibrium towards equilibrium, and satisfies a very restrictive stability requirement.

We believe that the new evolution equation will constitute an important mathematical "tool" for several, radically different nonequilibrium problems. Different schools of thought, with contrasting perspectives on the physical meaning of entropy and irreversibility, could all extract important insights from the richness of structure implied by this relevant nonlinear equation. Applications could also be found beyond the domain of physics, e.g., to time-dependent problems in Information Theory.

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8. Equilibrium solution ρ_e is stable according to Liapunoff if and only if for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that any solution $\rho(t)$ with $||\rho(0) - \rho_e|| < \delta(\varepsilon)$ remains with $||\rho(t) - \rho_e|| < \varepsilon$ for every $t > 0$, where $||\cdot||$ denotes the norm on $\mathcal{L}$ defined by $||A|| = (A|A)$.
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