Article

No-Signaling in Steepest Entropy Ascent: A Nonlinear, Non-Local, Non-Equilibrium Quantum Dynamics of Composite Systems Strongly Compatible with the Second Law

Rohit Kishan Ray [1,2,*] and Gian Paolo Beretta [3]

1   Department of Material Science and Engineering, Virginia Tech, Blacksburg, VA 24061, USA
2   Center for Theoretical Physics of Complex Systems, Institute for Basic Science (IBS), Daejeon 34126, Republic of Korea
3   Department of Mechanical and Industrial Engineering, University of Brescia, 25123 Brescia, Italy; gianpaolo.beretta@unibs.it
*   Correspondence: rkray@vt.edu

Abstract

Lindbladian formalism models open quantum systems using a 'bottom-up' approach, deriving linear dynamics from system–environment interactions. We present a 'top-down' approach starting with phenomenological constraints, focusing on a system's structure, subsystems' interactions, and environmental effects and often using a non-equilibrium variational principle designed to enforce strict thermodynamic consistency. However, incorporating the second law's requirement—that Gibbs states are the sole stable equilibria—necessitates nonlinear dynamics, challenging no-signaling principles in composite systems. We reintroduce 'local perception operators' and show that they allow to model signaling-free non-local effects. Using the steepest-entropy-ascent variational principle as an example, we demonstrate the validity of the 'top-down' approach for integrating quantum mechanics and thermodynamics in phenomenological models, with potential applications in quantum computing and resource theories.

Keywords: non-linear quantum thermodynamics; entropy production; no-signaling; steepest entropy ascent; local perception operators

1. Introduction

The study of open quantum systems broadly falls into two distinct approaches. The dominant method involves the Gorini–Kossakowski–Lindblad–Sudarshan (GKLS) master equation, typically under the assumption of weak coupling between a quantum system and its environment (for recent reviews, see [1,2]). This approach derives thermodynamically consistent reduced system dynamics starting from the Schrödinger equation combined system-environment evolution and subsequently tracing out environmental degrees of freedom [3–5]. We refer to this widely adopted strategy as the 'bottom-up' approach. Its effectiveness relies on idealized assumptions regarding the environment and the careful construction of appropriate Kraus operators, and it typically presumes that the initial state of the system-environment composite is factorizable.

In contrast, an alternative yet less explored methodology exists: the phenomenological or 'top-down' approach. This paradigm directly posits a general local dynamical law for the system's density operator based primarily on thermodynamic and phenomenological

considerations. It inherently accommodates strongly coupled subsystems and composite structures. Crucially, it explicitly incorporates the thermodynamic requirement that Gibbs states are the only conditionally stable equilibrium states [6,7], necessitating a departure from strictly linear evolution. Simmons and Park [6] demonstrated early on that nonlinearity is essential to achieve a consistent integration of quantum mechanics and thermodynamics. It is important to clarify here that our approach is distinct from the nonlinear quantum mechanics (NQM) recently subjected to stringent experimental tests [8,9]; rather, it represents a phenomenological route toward thermodynamically consistent, irreversible and non-equilibrium quantum dynamical models.

Recent experimental advances underscore the practical necessity for top-down phenomenological models, especially in scenarios where detailed environmental modeling is inaccessible or impractical [10]. Such models naturally describe 'apparent decoherence' stemming from unknown environmental interactions, employing minimal empirical parameters. Moreover, they conveniently handle situations where multiple conserved quantities, possibly non-commuting, are relevant at the local level [11].

Among the existing top-down formalisms, the Steepest Entropy Ascent (SEA) framework has emerged as particularly powerful and conceptually robust. Originating from the foundational works of Hatsopoulos and Gyftopoulos [12–15], and subsequently developed extensively by Beretta [16–24], SEA integrates thermodynamic irreversibility into quantum dynamics from first principles. This formalism has been successfully employed across diverse quantum systems [25–27], including quantum computational scenarios [28,29]. Nevertheless, despite its robustness and conceptual appeal, a rigorous demonstration of SEA's compatibility with fundamental quantum principles, particularly the no-signaling condition, has thus far remained underexplored [22–24,26].

This paper explicitly fills this critical gap. We rigorously prove that SEA dynamics inherently satisfy the no-signaling condition, even though its equations of motion incorporate nonlinearities and nonlocal structures. To this end, we introduce and utilize Local Perception Operators (LPOs), a conceptual tool that ensures subsystem locality in a nonlinear dynamical context. We establish the invariance of LPOs under local unitary transformations, which guarantees that no signaling between non-interacting subsystems emerges from the SEA dynamics. We further substantiate our claims through nontrivial illustrative examples. While the primary objective is to demonstrate that SEA-type dynamics respect no-signaling, we also take this opportunity to clarify the foundational motivation for invoking the SEA framework in the first place. In doing so, we show how it resolves subtle but important paradoxes that often go unaddressed in conventional bottom-up approaches.

The paper is structured as follows. Section 2 formally defines signaling and explores its relationship with nonlinearity. In Section 3, we revisit the philosophical motivations that led to the inception of SEA dynamics. Section 4 discusses conceptual issues surrounding quantum state individuality that nonlinear dynamics help resolve. Section 5 introduces a measure-theoretic representation for mixed ensembles, facilitating the rigorous handling of nonlinear dynamics. The Local Perception Operators (LPOs) are defined and analyzed in Section 6, and their critical role in establishing no-signaling is rigorously demonstrated in Section 7. The composite-system SEA equation is developed structurally in Section 8, followed by a variational derivation in Sections 9 and 10. Section 11 presents explicit numerical examples, and Section 12 summarizes our findings.

2. No-Signaling and Nonlinearity

Quantum mechanics (QM), as described by the Schrödinger-von Neumann formalism, is traditionally characterized by its linearity in state space and time evolution. Mean values of global and local properties and their time evolutions under unitary (and completely

positive) dynamics are linear functionals of the density operator. This linearity implies no-cloning, first shown by Park [30] and later rediscovered by Ghirardi [31], Wootters and Zurek [32], and Dieks [33]. Linearity also implies no-signaling [34–37]. However, Weinberg questioned this linear characteristic of QM [38]. He introduced nonlinearity into the operator formalism of the Schrödinger equation while preserving order-one homogeneity. This was one of the earliest examples of what is now broadly referred to as nonlinear quantum mechanics (NQM). Gisin [39] showed that such formalism could facilitate faster-than-light communication, viz., signaling. Polchinski [40] proposed an alternative method to prevent signaling by restricting local evolutions in order to depend solely on the corresponding reduced density operators. However, this formalism still allowed signaling between different branches of the wave function (Everett telephone) [40]. Wódkiewicz and Scully [41] analyzed the nonlinear evolution of two-level atoms, with solutions later shown to depend on interpretation [42]. Czachor also showed that a nonlinear operator induces mobility phenomena (non-conservation of the inner-product of two pure states) [43].

Nonlinearity introduced via stochastic QM through Lindblad operator formalism [44] for open quantum systems has been shown to respect no-signaling. Weinberg-inspired nonlinearities, interestingly, enable faster algorithms to solve *NP*-complete problems in polynomial time [45]. Later work [46] demonstrated that nonlinearity in QM can be incorporated without violating no-signaling, as long as time evolution is nonlinear while state space and operators remain linear. More recently, convex quasilinear maps [47,48] were shown to support nonlinear QM dynamics without signaling, preserving key features of QM. Rembieliński and Caban [47] identified this as the minimal allowable deviation from QM's linear structure. In a recent work, Kaplan and Rajendran [49] showed that, in a low-temperature limit, the non-linearity introduced in quantum field theory results in causal nonlinear quantum mechanics, which is also another version of NQM.

Therefore, introducing nonlinearity in quantum mechanics (QM) can lead to exotic interactions that are difficult to justify physically. Nonlinear theories, as inspired by the Weinberg approach, allow for entropy oscillation [43]. The stochastic, jump-induced mixing of pure states [44] offers a basis for developing a thermodynamically consistent nonlinear master equation for the density operator of an open system. However, the search for theoretically consistent nonlinear models of quantum thermodynamics remains an open challenge. This is where we claim—and later demonstrate in the following sections—that the 'top-down' approach of SEA resolves the signaling problem entirely (pictorially sketched in Figure 1). From a philosophical perspective, SEA also addresses other conceptual conundrums, in particular the Schrödinger-Park paradox [50,51], as was already discussed by Park [52] and Beretta [53].

The no-signaling condition, as noted in [46], is typically enforced by requiring that, in the absence of mutual interactions between subsystems A and B, the evolution of A's local observables depends solely on its reduced state. Formally, we express this as follows:

$$\frac{\mathrm{d}\rho_J}{\mathrm{d}t} = f(\rho_J), \tag{1}$$

where ρ_J is the reduced density operator (local state) of subsystem J. The SEA formalism, however, adopts a less restrictive perspective [22]. It requires that, if A and B are non-interacting, the law of evolution must not permit a local operation within B to influence the time evolution of A's local (reduced, marginal) state. Consequently, the condition $\rho_A = \rho'_A$, applied to the two different states $\rho \neq \rho_A \otimes \rho_B$ and $\rho' = \rho_A \otimes \rho_B$, does not necessitate $\mathrm{d}\rho_A/\mathrm{d}t = \mathrm{d}\rho'_A/\mathrm{d}t$. This is because the local evolution can still be influenced by past interactions, such as existing entanglement and correlations, without violating the no-signaling principle. This highlights two key ideas: (1) analyzing local evolutions

allows detection of correlations, but only those that can be classically communicated between subsystems, and (2) in the absence of interactions, nonlinear dynamics can cause correlations to diminish (spontaneous decoherence) but cannot create new correlations. We formally express this no-signaling condition as follows (see Section 8 for details),

$$\frac{\mathrm{d}\rho_J}{\mathrm{d}t} = f(\rho_J, (C_k)^J),$$

(2)

where $(C_k)^J$ are 'local perception operators' (LPO) (see Section 6 for their definition).
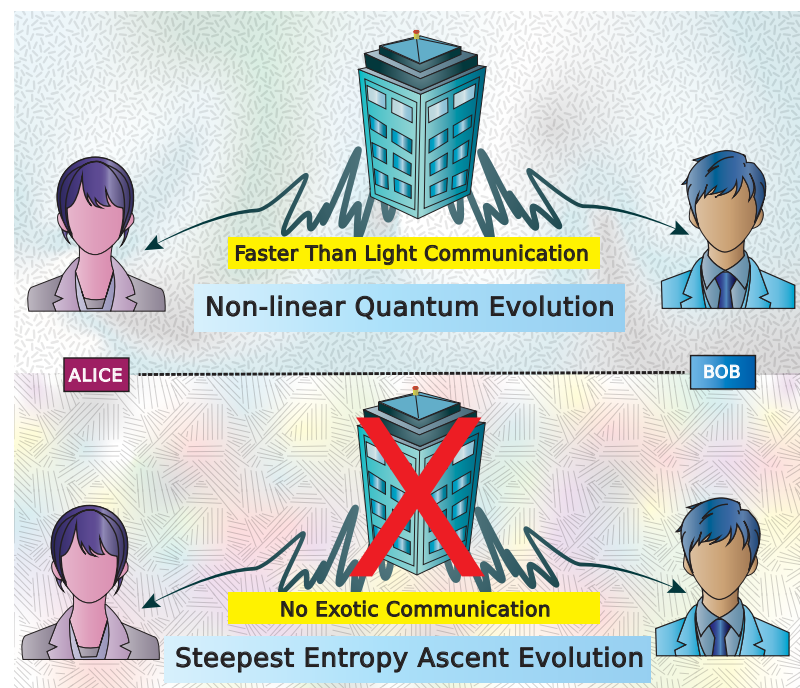

Figure 1. Schematic representation of no-signaling situation in the case of nonlinear quantum evolution. Alice and Bob are correlated but are non-interacting in a nonlinear quantum theory (we imply the nonlinearities via the texture of the embedding background in the schematic). In the top part, a non-linear quantum evolution as described by Polchinski [40] or Gisin [39] may entail faster-than-light communication in some form (denoted as the telephone booth; see text for details). In the bottom part, instead, the embedding nonlinear steepest-entropy-ascent evolution cannot establish a similar exotic communication.

3. Philosophical Motivations Leading to SEA

Philosophically, SEA evolution was designed to conceive the irreversible equilibration and decoherence as a fundamental spontaneous dynamical feature, contrary to the coarse-graining approach. Entropy, the second law, and irreversibility could attain a more fundamental status as deterministic outcomes of quantal evolution [54], without contradicting standard QM. From the outset [16,17], nonlinearity was recognized as essential [6,55] for this purpose. While the prevalent notion was—and still is—that the second law is emerging and statistical, the pioneers of SEA believed that the many 'knots of thermodynamics' [55,56] could, at least conceptually, be untied by elevating it to a more fundamental stature. This motivated Hatsopoulos and Gyftopoulos (HG) to develop a unified theory of mechanics and thermodynamics [12–15]. They laid the foundation for what is now known as resource theory in quantum thermodynamics by extending ontological quantum states to include both pure and mixed density operators. The HG unified (resource) theory explicitly defines and fully characterizes the concept—later rediscovered as 'ergotropy' [57]—of

maximum work extractable in a unitary process with cyclic parameter changes (theorems 3.2 and 3.3 in [13], which pioneered also the use of 'majorization' in quantum theory).

Beretta's doctoral thesis [58] aimed to complete the HG theory with two key goals: (1) to design an equation of motion that establishes the second law of thermodynamics as a theorem, ensuring Gibbs states are the only conditionally stable equilibrium states; and (2) to develop an unambiguous statistical representation of heterogeneous ensembles formed by the statistical mixing of homogeneous (pure) ensembles as one-to-one with the full set of pure and mixed density operators.

The outcome of the first objective was the SEA formalism, including the introduction of local perception operators. SEA formalism proved to be both robust and versatile, making it a powerful tool for thermodynamically consistent non-equilibrium modeling. Its applications expanded beyond its original scope to include phenomenological models in areas such as decoherence [59], mesoscopic transport rheology [60,61], phase transitions [62], and quantum computing [27–29,63].

The second objective led to the representation of generic ensembles using measure-theoretic distributions over the set of ontological states. In the context of quantum thermodynamics, an "unambiguous statistical representation of heterogeneous ensembles" ensures a clear distinction between intrinsic quantum uncertainties and classical statistical mixing. This approach—detailed in Section 5 following [53,58]—represents heterogeneous ensembles as distributions over the full set of pure and mixed density operators, thus avoiding ambiguities associated with von Neumann's statistical interpretation and resolving the Schrödinger-Park paradox.

4. Nonlinear Dynamics and the Ontic State Conundrum

From a foundational perspective, the HG approach of assigning ontological status to the full set of density operators, pure and mixed, offers a potential resolution to the Schrödinger–Park paradox concerning the ambiguity of individual quantum states [50,51,53]. As noted by Park [52], treating density operators as one-to-one with homogeneous ensembles resolves ambiguity under two conditions: (1) that they evolve unitarily for short times, with nonlinear dynamics dominating over longer periods; and (2) that a full tomography, even for heterogeneous preparations, must include both linear and nonlinear observables, such as entropy, adiabatic availability, ergotropy, and free energy.

The von Neumann prescription [64] (Chapter III) assigns a density operator to a statistical mixture of pure states, with mixing coefficients reflecting ignorance. However, Schrödinger [50] questioned this from the outset due to the non-unique decomposition of a mixed density operator into weighted sums of pure states. von Neumann's approach irrecoverably blends quantal probabilities with non-quantal statistical weights, creating the ambiguity highlighted by Schrödinger and Park. To understand how a law of nonlinear evolution may resolve this issue, let $\hat{\Pi}_1^0$ and $\hat{\Pi}_2^0$ be two distinct homogeneous preparations. At time $t = 0$, they generate two initial ensembles of identically prepared, strictly isolated, and non-interacting systems. The 'hat' on Π denotes the homogeneity of the preparation, and superscript '0' refers to time $t = 0$. According to the traditional statistical formulation by von Neumann, a preparation, $\hat{\Pi}$, is *homogeneous* if and only if it cannot be replicated using any statistical composition of two *different* preparations. The statistics of measurement outcomes from a homogeneous preparation are represented as a pure (idempotent) density operator, $\hat{\rho}$, satisfying $\hat{\rho} = \hat{\rho}^2$. Consider a third (heterogeneous) preparation, Π_3^0, obtained via the statistical mixture of $\hat{\Pi}_1^0$ and $\hat{\Pi}_2^0$ with a fixed weight, $0 < w < 1$, so that, formally, we may write $\Pi_3^0 = w\hat{\Pi}_1^0 + (1-w)\hat{\Pi}_2^0$. To obtain quantum statistical mechanics (QSM) consistent with Gibbs–Boltzmann, Fermi–Dirac, and Bose–Einstein equilibrium distributions, von Neumann postulated that a heterogeneous preparation, such as Π_3^0, should be

represented as a mixed density operator, $\rho_3^0 = w\,\hat{\rho}_1^0 + (1-w)\,\hat{\rho}_2^0$. This formulation, partly due to the linearity of quantum mechanics, has served as the foundation for successful frameworks such as equilibrium QSM and the Lindbladian theory of open systems.

These successes often lead us to overlook the Schrödinger–Park paradox or accept it as a seemingly enigmatic aspect of quantum theory. It arises when interpreting a heterogeneous ensemble. If we assume that each member of the ensemble is prepared in either state $\hat{\rho}_1^0$ (with probability w) or $\hat{\rho}_2^0$ (with probability $1-w$), excluding all other states, we face a contradiction. The same mixed density operator, ρ_3^0, allows infinitely many alternative decompositions. For instance, the decomposition $\rho_3^0 = w'\,\hat{\rho}_4^0 + (1-w')\,\hat{\rho}_5^0$ suggests that the ensemble consists only of $\hat{\rho}_4^0$ and $\hat{\rho}_5^0$, excluding $\hat{\rho}_1^0$ and $\hat{\rho}_2^0$. This inconsistency challenges the notion of individual states in a heterogeneous ensemble and complicates the interpretation of QSM. Under a linear evolution map for the homogeneous preparations, $\hat{\Pi}^t = \mathcal{L}_t(\hat{\Pi}^0)$, and mixing of homogeneous preparations with time-invariant weights, we have

$$\Pi_3^t = w\,\hat{\Pi}_1^t + (1-w)\,\hat{\Pi}_2^t = w\,\mathcal{L}_t(\hat{\Pi}_1^0) + (1-w)\,\mathcal{L}_t(\Pi_2^0) \tag{3}$$

and, via the linearity of $\mathcal{L}_t$, $\Pi_3^t = \mathcal{L}_t(\hat{\Pi}_3^0)$. Thus, the von Neumann recipe gives $\rho_3^t = w\,\hat{\rho}_1^t + (1-w)\,\hat{\rho}_2^t$ and, for linear observables, it yields the correct statistics of measurements,

$$\text{Tr}\big(A\,\rho_3^t\big) = w\,\text{Tr}\big(A\,\hat{\rho}_1^t\big) + (1-w)\,\text{Tr}\big(A\,\hat{\rho}_2^t\big). \tag{4}$$

It also entails that, for an N-level system, a full tomography of a heterogeneous preparation requires the measurement of a quorum of only N^2-1 independent linear observables.

Instead, if the evolution map for homogeneous preparations is nonlinear, $\hat{\Pi}^t = \mathcal{N}_t(\hat{\Pi}^0)$, and the statistical mixing weights remain time-invariant, we obtain

$$\Pi_3^t = w\,\hat{\Pi}_1^t + (1-w)\,\hat{\Pi}_2^t = w\,\mathcal{N}_t(\hat{\Pi}_1^0) + (1-w)\,\mathcal{N}_t(\Pi_2^0) \tag{5}$$

and, in general, due to the nonlinearity of $\mathcal{N}_t$, this does not satisfy $\Pi_3^t \neq \mathcal{N}_t(\hat{\Pi}_3^0)$. This contradicts the von Neumann prescription, which would assign the density operator $\rho_3^t = w\,\rho_1^t + (1-w)\,\rho_2^t$ to preparation Π_3^t and $\rho_3^0 = w\,\rho_1^0 + (1-w)\,\rho_2^0$ to preparation Π_3^0. As noted by Rembieliński and Caban [47], a sufficient condition for a nonlinear map to be no-signaling is convex-quasilinearity. That is, it must always admit a w', with $0 \leq w' \leq 1$, such that $\rho_3^t = w'\,\rho_1^t + (1-w')\,\rho_2^t$. Ferrero et al. [46] made a similar observation but left its interpretation as an open problem.

The HG ontological hypothesis alleviates this interpretation issue. Regardless of how a density operator is decomposed, w and $1-w$ no longer represent epistemic ignorance. This is because both pure and mixed density operators have ontic status, meaning they represent homogeneous preparations.

5. Measure-Theoretic Representation of Statistics from Mixed Ensembles

To resolve the Schrödinger-Park paradox about individual states in QSM without contradicting the successes of QM, Beretta [58] proposed replacing the von Neumann prescription for measurement statistics from mixed ensembles with an unambiguous representation akin to classical statistical mechanics. The recipe assigns to every preparation Π (whether homogeneous or heterogeneous) a normalized *measure* μ_Π defined on the (ontic) quantal state domain $\mathcal{P}$, which consists of the mathematical objects representing homogeneous preparations (homogeneous ensembles). The normalization condition is

$$\mu_\Pi(\mathcal{P}) = \int_\mathcal{P} \mu_\Pi(d\rho) = 1. \tag{6}$$

In usual von Neumann QSM, the quantal state domain $\mathcal{P}_{\text{QSM}}$ is the set of all one-dimensional projection operators (i.e., the idempotent density operators) defined on the Hilbert space of the system. Instead, the HG-unified theory assumes the broader quantal state domain $\mathcal{P}_{\text{HG}}$ consisting of the set of all possible density operators ρ, idempotent and not.

Among the measures defined on $\mathcal{P}$, the *Dirac measures*, defined as follows, play a crucial role. Let $\mathcal{E}$ denote any subset of $\mathcal{P}$, and then

$$\delta_\rho(\mathcal{E}) = \begin{cases} 1, & \text{if } \rho \in \mathcal{E} \\ 0, & \text{if } \rho \notin \mathcal{E} \end{cases} \tag{7}$$

The support of a Dirac measure, $\delta_{\tilde{\rho}}$, i.e., the subset of $\mathcal{P}$ for which it is nonzero, is a single point coinciding with the state operator $\tilde{\rho}$.

The statistical mixing of preparations, $\Pi_3 = w\,\Pi_1 + (1-w)\,\Pi_2$, is represented by the weighted sum of the corresponding measures,

$$\mu_{\Pi_3} = w\,\mu_{\Pi_1} + (1-w)\,\mu_{\Pi_2}. \tag{8}$$

This representation removes ambiguities because of the following:

1. No Dirac measure can be decomposed into a weighted sum of different Dirac measures, satisfying von Neumann's definition of homogeneous preparations.
2. Any measure has a unique decomposition into a weighted sum (or integral) of Dirac measures—removing the Schrödinger–Park paradox. This ensures that each member of a heterogeneous ensemble is in some well defined (albeit unknown) individual state.

The expected mean value of any physical observable represented by the (linear or nonlinear) functional g defined on $\mathcal{P}$ is given for preparation $\Pi_3 = w\,\Pi_1 + (1-w)\,\Pi_2$ by

$$\begin{aligned} \overline{\langle g \rangle}_{\Pi_3} &= \int_{\mathcal{P}} g(\rho)\,\mu_{\Pi_3}(\mathrm{d}\rho) \\ &= \int_{\mathcal{P}} g(\rho)\,[w\,\mu_{\Pi_1}(\mathrm{d}\rho) + (1-w)\,\mu_{\Pi_2}(\mathrm{d}\rho)] \\ &= w\int_{\mathcal{P}} g(\rho)\,\mu_{\Pi_1}(\mathrm{d}\rho) + (1-w)\int_{\mathcal{P}} g(\rho)\,\mu_{\Pi_2}(\mathrm{d}\rho). \end{aligned} \tag{9}$$

If the component preparations are homogeneous, i.e., if $\mu_{\hat{\Pi}_1} = \delta_{\rho_1}$ and $\mu_{\hat{\Pi}_2} = \delta_{\rho_2}$, then

$$\begin{aligned} \overline{\langle g \rangle}_{\Pi_3} &= w\int_{\mathcal{P}} g(\rho)\,\delta_{\rho_1}(\mathrm{d}\rho) + (1-w)\int_{\mathcal{P}} g(\rho)\,\delta_{\rho_2}(\mathrm{d}\rho) \\ &= w\,g(\rho_1) + (1-w)\,g(\rho_2) \\ &= w\,\langle g \rangle_{\hat{\Pi}_1} + (1-w)\,\langle g \rangle_{\hat{\Pi}_2}. \end{aligned} \tag{10}$$

For example, for $\mathcal{P} = \mathcal{P}_{\text{HG}}$ and $g(\rho) = -k_{\text{B}}\,\text{Tr}(\rho \ln \rho)$, this gives the 'proper' expected value of measurements of the von Neumann entropy, i.e., the weighted sum of the entropies of the component homogeneous sub-ensembles.

It is noteworthy that, for observables represented by linear functionals on $\mathcal{P}$, such as $g(\rho) = \text{Tr}(G\rho)$, we have

$$\begin{aligned} \overline{\langle g \rangle}_{\Pi_3} &= w\,\text{Tr}\left(G\,\rho_{\hat{\Pi}_1}\right) + (1-w)\,\text{Tr}\left(G\,\rho_{\hat{\Pi}_2}\right) \\ &= \text{Tr}\left(G\,[w\,\rho_{\hat{\Pi}_1} + (1-w)\,\rho_{\hat{\Pi}_2}]\right) \\ &= \text{Tr}(G\,W_{\Pi_3}) = g(W_{\Pi_3}), \end{aligned} \tag{11}$$

where W_{Π_3} is the usual von Neumann statistical operator, defined by the weighted sum of the density operators representing the component homogeneous sub-ensembles,

$$W_{\Pi_3} = w\,\rho_{\hat{\Pi}_1} + (1-w)\,\rho_{\hat{\Pi}_2}. \tag{12}$$

W_{Π_3} is clearly an element of the convex completion, $\mathcal{C}(\mathcal{P})$, of the quantal state domain $\mathcal{P}$. Notice that, in general $\mathcal{C}(\mathcal{P}_{\mathrm{QSM}}) = \mathcal{C}(\mathcal{P}_{\mathrm{HG}}) = \mathcal{P}_{\mathrm{HG}}$, and for a qubit $\mathcal{P}_{\mathrm{QSM}}$ and $\mathcal{P}_{\mathrm{HG}}$ map to the Bloch sphere and the Bloch ball, respectively. An evaluation of the linear functional $g(\cdot) = \mathrm{Tr}(G\cdot)$ at W_{Π_3} provides the correct expected mean value of measurements of the corresponding observable on the heterogeneous ensemble.

But W_{Π_3} does not fully describe the heterogeneous preparation Π_3, because different preparations can yield the same linear tomography. In other words, the linear tomography—obtained by measuring the expected mean values $\overline{\langle q_j \rangle}$ of a quorum of N^2-1 independent linear observables Q_j and solving $\overline{\langle q_j \rangle} = \mathrm{Tr}(Q_j\,W)$ for W—is insufficient to characterize a preparation. A decomposition of W into different weighted sums of density operators has no meaning in this theory, but it emphasizes that different heterogeneous preparations may result in the same linear tomography. Even within orthodox QSM ($\mathcal{P} = \mathcal{P}_{\mathrm{QSM}}$), resolving the intrinsic quantum probabilities of homogeneous preparations from the extrinsic uncertainties of mixing requires additional independent information beyond linear tomography.

The measure-theoretic description of preparations enables a statistical quantum theory in which both linear and nonlinear functionals of the density operator correspond to independently and directly measurable properties of a quantum system, such as entropy, ergotropy, and adiabatic availability. A nonlinear evolution equation for the density operators in $\mathcal{P}$, i.e., for the homogeneous preparations, may provide additional nonlinear observables by measuring linear and nonlinear observables at different times. As already realized by Park [52], nonlinearity holds the promise of preserving and reintegrating the notion of an individual state into quantum theory.

6. Local Perception Operators (LPOs)

In linear QM, the system's composition is specified by declaring the following:

1. The Hilbert space structure as the direct product $\mathcal{H} = \otimes_{J=1}^{M} \mathcal{H}_J$ of the subspaces of the M component subsystems.
2. The overall Hamiltonian operator $H = \sum_{J=1}^{M} H_J \otimes I_{\bar{J}} + V$, where H_J (on $\mathcal{H}_J$) is the local Hamiltonian of the J-th subsystem, $I_{\bar{J}}$ is the identity on the direct product $\mathcal{H}_{\bar{J}} = \otimes_{K \neq J} \mathcal{H}_K$ of all the other subspaces, and V (on $\mathcal{H}$) is the interaction Hamiltonian.

The linear von Neumann law of evolution, $\dot{\rho} = -\mathrm{i}[H, \rho]/\hbar$, has a universal structure and involves local evolutions through partial tracing,

$$\dot{\rho}_J = -\frac{\mathrm{i}}{\hbar}[H_J, \rho_J] - \frac{\mathrm{i}}{\hbar}\,\mathrm{Tr}_{\bar{J}}([V, \rho]). \tag{13}$$

Thus, we recover the universal law $\dot{\rho}_J = -\mathrm{i}[H_J, \rho_J]/\hbar$ for the local density operator $\rho_J = \mathrm{Tr}_{\bar{J}}(\rho)$ if the subsystem J does not interact with the others (i.e., if $V = I_J \otimes V_{\bar{J}}$).

Instead, a fully nonlinear evolution equation for the density operator cannot have a universal structure because the subdivision into subsystems must be explicitly embedded into the structure of the dynamical law (see [24] for more on this). A different subdivision requires a different equation of motion. The complex structure of the SEA evolution law reflects the cost of abandoning linearity but ensures compatibility with the crucial constraint that correlations should not build up, and signaling should not occur between subsystems, other than via the interaction Hamiltonian V through the standard Schrödinger term $-\mathrm{i}[H, \rho]/\hbar$ in the evolution law.

Seldom used in composite quantum dynamics but crucial, in our opinion, are the physical observables first introduced in [17] and later (starting with [19]) referred to as 'local perceptions of global observables'. These are represented via the 'local perception operators' (LPO) on $\mathcal{H}_J$, defined along with their 'deviation from the local mean value' operators and covariance functionals as follows:

$$(X)^J_\rho = \mathrm{Tr}_{\bar{J}}[(I_J \otimes \rho_{\bar{J}})X], \tag{14}$$

$$\Delta(X)^J_\rho = (X)^J_\rho - I_J\,\mathrm{Tr}\left[\rho_J(X)^J_\rho\right], \tag{15}$$

$$(X,Y)^J_\rho = \tfrac{1}{2}\,\mathrm{Tr}\left[\rho_J\left\{\Delta(X)^J_\rho, \Delta(Y)^J_\rho\right\}\right], \tag{16}$$

where $\rho_{\bar{J}} = \mathrm{Tr}_J(\rho)$. For a bipartite system, AB, the LPOs $(X)^A_\rho$ (on $\mathcal{H}_A$) and $(X)^B_\rho$ (on $\mathcal{H}_B$) are the unique operators that, for a given X on $\mathcal{H}_{AB}$ and for all states ρ, satisfy the identity

$$\mathrm{Tr}\left[\rho_A(X)^A_\rho\right] = \mathrm{Tr}[(\rho_A \otimes \rho_B)X] = \mathrm{Tr}\left[\rho_B(X)^B_\rho\right]. \tag{17}$$

This confirms that they encapsulate all the information that A and B can infer about the global observable X by classically sharing their local states.

Operator $(X)^A_\rho$ can be viewed as the projection onto $\mathcal{H}_A$ of the operator X weighted according to the local state ρ_B of subsystem B. However, it is a local observable for subsystem A, which depends on the overall state, ρ, and the overall observable, X. Its local mean value, $\mathrm{Tr}\left[\rho_A(X)^A_\rho\right]$, differs from the mean value, $\mathrm{Tr}(\rho X)$, for the overall system, AB, except when A and B are uncorrelated ($\rho = \rho_A \otimes \rho_B$). It was dubbed 'local perception' because, even if B performs a local tomography and sends the measured ρ_B to A via classical communication, the most that A can measure locally about the overall observable X is $(X)^A_\rho$.

The overall energy and entropy of the composite system are locally perceived within subsystem J through the operators $(H)^J_\rho$ and $(S)^J_\rho$ defined on $\mathcal{H}_J$ by Equation (14), respectively, with $X = H$, the overall Hamiltonian, and $X = S(\rho)$, the overall (non-negative) entropy operator defined by

$$S(\rho) = -k_{\mathrm{B}}\mathrm{Bln}(\rho), \tag{18}$$

where we define the discontinuous log function

$$\mathrm{Bln}(x) = \begin{cases} \ln(x), & \text{for } 0 < x \le 1, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Note that the 'locally perceived overall entropy' operator

$$(S)^J_\rho = -k_{\mathrm{B}}\,\mathrm{Tr}_{\bar{J}}[(I_J \otimes \rho_{\bar{J}})\mathrm{Bln}(\rho)], \tag{20}$$

is different from the 'local entropy' operator

$$S(\rho_J) = -k_{\mathrm{B}}\mathrm{Bln}(\rho_J). \tag{21}$$

Its local mean value, $\mathrm{Tr}\left[\rho_J(S)^J_\rho\right] = -k_{\mathrm{B}}\,\mathrm{Tr}\left[(\rho_J \otimes \rho_{\bar{J}})\mathrm{Bln}(\rho)\right]$, is different from the local entropy $\mathrm{Tr}\left[\rho_J S(\rho_J)\right] = -k_{\mathrm{B}}\,\mathrm{Tr}\left[\rho_J \ln(\rho_J)\right]$. Only when $\rho = \rho_J \otimes \rho_{\bar{J}}$ are they related via

$$\mathrm{Tr}\left[\rho_J(S)^J_\rho\right] = \mathrm{Tr}\left[\rho_J S(\rho_J)\right] + \mathrm{Tr}\left[\rho_{\bar{J}} S(\rho_{\bar{J}})\right]$$
$$= -k_{\mathrm{B}}\,\mathrm{Tr}[\rho \ln(\rho)]. \tag{22}$$

Likewise, the 'locally perceived overall Hamiltonian' operator $(H)_\rho^J$ differs from the 'local Hamiltonian' operator H_J. Its local mean value, $\mathrm{Tr}\left[\rho_J(H)_\rho^J\right] = \mathrm{Tr}\left[(\rho_J\otimes\rho_{\bar{J}})H\right]$, is different from the local mean energy, $\mathrm{Tr}\left(\rho_J H_J\right)$, and only when $V = I_J\otimes V_{\bar{J}}$ are they related via

$$\mathrm{Tr}\left[\rho_J(H)_\rho^J\right] = \mathrm{Tr}\left(\rho_J H_J\right) + \mathrm{Tr}\left(\rho_{\bar{J}}H_{\bar{J}}\right) = \mathrm{Tr}(\rho H). \tag{23}$$

However, it is noteworthy that, when the overall observable, X is 'separable for subsystem J,' in the sense that $X = X_J\otimes I_{\bar{J}} + I_J\otimes X_{\bar{J}}$, then, even if $\rho \neq \rho_J\otimes\rho_{\bar{J}}$, the deviations and covariances reduce to their local versions,

$$\Delta(X)_\rho^J = \Delta X_J = X_J - I_J\,\mathrm{Tr}\left[\rho_J X_J\right], \tag{24}$$

$$(X, Y)_\rho^J = \tfrac{1}{2}\,\mathrm{Tr}\left[\rho_J\left\{\Delta X_J, \Delta Y_J\right\}\right]. \tag{25}$$

In general, the mean interaction energy, $\mathrm{Tr}\left(\rho_J V_{J\bar{J}}\right)$, and the mutual information, $\mathrm{Tr}\left(\rho_J\mu_{J\bar{J}}\right)$, between subsystem J and the rest of the system, $\bar{J}$, are given via the respective mean values of the following global operators:

$$V_{J\bar{J}} = H - H_J\otimes I_{\bar{J}} - I_J\otimes H_{\bar{J}}, \tag{26}$$

$$\mu_{J\bar{J}} = \mathrm{Bln}(\rho) - \mathrm{Bln}(\rho_J)\otimes I_{\bar{J}} - I_J\otimes\mathrm{Bln}(\rho_{\bar{J}}), \tag{27}$$

whose LPOs satisfy the identities

$$\begin{aligned}
\mathrm{Tr}&\left[\rho_J(H)_\rho^J\right] - \mathrm{Tr}\left[\rho_J(V_{J\bar{J}})_\rho^J\right] \\
&= \mathrm{Tr}\left(\rho_J H_J\right) + \mathrm{Tr}\left(\rho_{\bar{J}}H_{\bar{J}}\right) \\
&= \mathrm{Tr}(\rho H) - \mathrm{Tr}\left(\rho_J V_{J\bar{J}}\right),
\end{aligned} \tag{28}$$

$$\begin{aligned}
\mathrm{Tr}&\left[\rho_J(S)_\rho^J\right] + k_{\mathrm{B}}\,\mathrm{Tr}\left[\rho_J(\mu_{J\bar{J}})_\rho^J\right] \\
&= -k_{\mathrm{B}}\,\mathrm{Tr}\left[\rho_J\ln\left(\rho_J\right)\right] - k_{\mathrm{B}}\,\mathrm{Tr}\left[\rho_{\bar{J}}\ln\left(\rho_{\bar{J}}\right)\right] \\
&= -k_{\mathrm{B}}\,\mathrm{Tr}[\rho\ln(\rho)] + k_{\mathrm{B}}\,\mathrm{Tr}\left(\rho_J\mu_{J\bar{J}}\right).
\end{aligned} \tag{29}$$

7. No-Signaling and LPOs

To formalize the no-signaling definition following [22], as discussed above, we adopt the view that local operations can be acted on a subsystem, $\bar{J}$, only by means of a controlled time dependence of its local Hamiltonian operator, $H_{\bar{J}}$, or its interaction operator, $V_{\bar{J}B}$, with some other subsystem, B (possibly a properly modeled environment or heat bath). Hence, we adopt the following:

Definition 1 (No-signaling). *If J and $\bar{J}$ are non-interacting, no local aspect of the time evolution of J can be affected by local unitary operations acted on $\bar{J}$, nor by other aspects of the local time evolution of $\bar{J}$.*

Accordingly, consider a composite, AB, in state ρ. A local unitary operation on B changes the state to

$$\rho' = (I_A\otimes U_B)\,\rho\,(I_A\otimes U_B^\dagger), \tag{30}$$

where U_B is an arbitrary unitary operator ($U_B^\dagger U_B = I_B$). Using the properties of the partial trace, in particular (Z_{AB} is a generic composite system operator),

$$\mathrm{Tr}_B[(I_A \otimes X_B)Z_{AB}] = \mathrm{Tr}_B[Z_{AB}(I_A \otimes X_B)], \tag{31}$$

$$\mathrm{Tr}_A[(I_A \otimes X_B)Z_{AB}(I_A \otimes Y_B)] = X_B \, \mathrm{Tr}_A(Z_{AB})Y_B, \tag{32}$$

we obtain the identities

$$\rho_B = \mathrm{Tr}_A[(I_A \otimes U_B^\dagger)\rho'(I_A \otimes U_B)] = U_B^\dagger \rho_B' U_B, \tag{33}$$

$$\begin{aligned}
\rho_A' &= \mathrm{Tr}_B[(I_A \otimes U_B)\rho(I_A \otimes U_B^\dagger)] \\
&= \mathrm{Tr}_B[(I_A \otimes U_B^\dagger U_B)\rho] \\
&= \mathrm{Tr}_B[(I_A \otimes I_B)\rho] = \rho_A,
\end{aligned} \tag{34}$$

which confirms that a local operation on B does not affect the local state, ρ_A, of A. This result supports the usual idea [46] that, for no-signaling, it is sufficient that the dynamical model implies evolutions of local observables that depend only on ρ_A. But it is seldom noted that this is not a necessary condition.

Next, we prove that not only the local reduced state ρ_A but also the LPO $(F(\rho))^A$ of any well-defined nonlinear function, $F(\rho)$, of the overall state (such as the entropy function $S(\rho)$ defined earlier) remains unaffected by local unitary operations on B, as per Equation (30). Since the SEA formalism is based on these local perception operators, this result is an important lemma in the proof that SEA is no-signaling.

So, let us apply Equation (30) to a generic observable represented by a linear operator on $\mathcal{H}$ that we denote as $F(\rho)$, whether it is a nontrivial function of ρ, such as $S(\rho)$, or not a function of ρ, such as H. The corresponding LPO for subsystem A, according to the defining Equation (14), is given by

$$(F(\rho))^A = \mathrm{Tr}_B[(I_A \otimes \rho_B)F(\rho)]. \tag{35}$$

A function of ρ is defined from its eigen-decomposition via

$$F(\rho) = \Lambda F(D)\Lambda^\dagger = \sum_j F(\lambda_j)|\lambda_j\rangle\langle\lambda_j|, \tag{36}$$

where $\rho = \Lambda D\Lambda^\dagger$, $D = \sum_j \lambda_j|j\rangle\langle j|$, and $\Lambda = \sum_j|\lambda_j\rangle\langle j|$. Since unitary transformations do not alter the eigenvalues,

$$F(\rho') = \Lambda' F(D)\Lambda'^\dagger \text{ where } \Lambda' = (I_A \otimes U_B)\Lambda. \tag{37}$$

Therefore, using Equation (33) in the last step, we obtain

$$\begin{aligned}
(F(\rho'))^A &= \mathrm{Tr}_B[(I_A \otimes \rho_B')F(\rho')] \\
&= \mathrm{Tr}_B[(I_A \otimes \rho_B')(I_A \otimes U_B)\Lambda F(D)\Lambda^\dagger(I_A \otimes U_B^\dagger)] \\
&= \mathrm{Tr}_B[(I_A \otimes U_B^\dagger \rho_B' U_B)\Lambda F(D)\Lambda^\dagger] \\
&= \mathrm{Tr}_B[(I_A \otimes \rho_B)F(\rho)] = (F(\rho))^A.
\end{aligned} \tag{38}$$

This confirms that local unitary operations on B do not affect the LPOs of A. Hence, the proper use of LPOs in a nonlinear evolution equation does not cause signaling issues.

8. The Local Structure of Dissipation in Composite-System SEA Dynamics

We are now ready to introduce the final but essential ingredient of a general composite-system nonlinear QM or master equation. This involves the system's structure-dependent expressions, which determine how each subsystem contributes separately to the dissipative term in the equation of motion for the overall state, ρ. As discussed above—and recognized in early SEA literature [17,22,24]—the composite-system nonlinear evolution must explicitly reflect the internal structure of the system. This requires declaring which subsystems are to be protected from nonphysical effects such as signaling, the exchange of energy, or the build-up of correlations between non-interacting subsystems. Using the notation introduced earlier, the structure proposed in [17,24] for the dissipative term, which supplements the usual Hamiltonian term, is given by

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H,\rho] - \sum_{J=1}^{M} \{\mathcal{D}_\rho^J, \rho_J\} \otimes \rho_{\bar{J}}, \tag{39}$$

where the 'local dissipation operators' $\mathcal{D}_\rho^J$ (on $\mathcal{H}_J$) may be nonlinear functions of the local observables of J, the reduced state ρ_J, and the local perception operators of overall observables. For the dissipative term to preserve $\mathrm{Tr}(\rho)$, operators $\{\mathcal{D}_\rho^J, \rho_J\}$ must be traceless. To preserve $\mathrm{Tr}(\rho H)$ [and possibly other conserved properties or charges, $\mathrm{Tr}(\rho C_k)$], operators $\{\mathcal{D}_\rho^J, \rho_J\}(H)_\rho^J$ [and $\{\mathcal{D}_\rho^J, \rho_J\}(C_k)_\rho^J$] must also be traceless. The rate of change of the overall system entropy $s(\rho) = -k_{\mathrm{B}}\,\mathrm{Tr}[\rho\ln(\rho)]$ is

$$\frac{\mathrm{d}s(\rho)}{\mathrm{d}t} = -\sum_{J=1}^{M} \mathrm{Tr}\left[\{\mathcal{D}_\rho^J, \rho_J\}(S)_\rho^J\right]. \tag{40}$$

The local nonlinear evolution of subsystem J is obtained via partial tracing over $\mathcal{H}_{\bar{J}}$, i.e., in general,

$$\frac{\mathrm{d}\rho_J}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H_J, \rho_J] - \frac{\mathrm{i}}{\hbar}\mathrm{Tr}_{\bar{J}}([V,\rho]) - \{\mathcal{D}_\rho^J, \rho_J\}, \tag{41}$$

where we recall that the second term in the RHS can be expressed, for weak interactions and under well-known assumptions, in Kossakowski–Lindblad form.

Before introducing the SEA assumption, as promised after Equation (2), we note that, for all possible choices of $\mathcal{D}_\rho^J$, Equation (39) defines a broad class of no-signaling nonlinear evolution equations. These form a broader class of nonlinear laws that are not restricted by the sufficient but not necessary condition that $\mathrm{d}\rho_J/\mathrm{d}t$ be a function of ρ_J only.

Finally, one way to introduce the SEA assumption in the spirit of the fourth law of thermodynamics [65–68], i.e., to implement the maximum entropy production principle (MEPP) in the present context [69], is to employ a variational principle.

9. General Composite System Version of the SEA Variational Principle

For the purpose of this section, we adopt the formalism developed for this context in [17,20,24]. To easily impose the constraints of preservation of non-negativity and self-adjointness of ρ during its time evolution, we define the generalized square root of ρ_J, $\gamma_J(t) = \sqrt{\rho_J(t)}\,U$ (U any unitary operator), so that

$$\rho_J = \gamma_J \gamma_J^\dagger. \tag{42}$$

The dissipative term in Equation (41) is rewritten as

$$-\{\mathcal{D}_\rho^J, \rho_J\} = \dot{\gamma}_J^d \gamma_J^\dagger + \gamma_J \dot{\gamma}_J^{d\dagger} \tag{43}$$

$$\text{with } \dot{\gamma}_J^d = -\mathcal{D}_\rho^J \gamma_J. \tag{44}$$

Next, on the set $\mathcal{L}(\mathcal{H}_J)$ of linear operators on $\mathcal{H}_J$, we define the real inner product, $(\cdot|\cdot)$,

$$(X|Y) = \text{Tr}\left(X^\dagger Y + Y^\dagger X\right)/2, \tag{45}$$

so that the unit trace condition for ρ_J is rewritten as $(\gamma_J|\gamma_J) = 1$, implying that the γ_J's lie on the unit sphere in $\mathcal{L}(\mathcal{H}_J)$. Their time evolutions, $\gamma_J(t)$, follow trajectories on this sphere. Along these trajectories, we can express the distance traveled between t and $t + dt$ as

$$d\ell_J = \sqrt{(\dot{\gamma}_J| \hat{G}_J(\gamma_J) |\dot{\gamma}_J)}\, dt, \tag{46}$$

where $\hat{G}_J(\gamma_J)$ is some real, dimensionless, symmetric, and positive–definite operator on $\mathcal{L}(\mathcal{H}_J)$ (superoperator on $\mathcal{H}$) that plays the role of a local metric tensor field (and may be a nonlinear function of γ_J).

The rates of change of the overall system entropy, $s(\rho)$, Equation (40), and of the overall system mean value of conserved properties $c_k(\rho) = \text{Tr}(\rho C_k)$, where $[C_k, H] = 0$, can be rewritten as

$$\frac{ds(\rho)}{dt} = \sum_{J=1}^M \dot{s}|_J \qquad \dot{s}|_J = \left(2(S)_\rho^J \gamma_J \middle| \dot{\gamma}_J^d\right), \tag{47}$$

$$\frac{dc_k(\rho)}{dt} = \sum_{J=1}^M \dot{c}_k|_J \qquad \dot{c}_k|_J = \left(2(C_k)_\rho^J \gamma_J \middle| \dot{\gamma}_J^d\right), \tag{48}$$

exhibiting additive contributions from the subsystems.

Finally, we state the variational principle that leads to expressions for the $\dot{\gamma}_J^d$ and the $\mathcal{D}_\rho^J$ that define the composite-system version of the SEA equation of motion. The time evolution ensures that the 'direction of change' of the local trajectory $\gamma_J(t)$, influenced by the dissipative part of the dynamics, maximizes the local contribution, $\dot{s}|_J$, to the overall system's entropy production rate. This occurs under the constraints $\dot{c}_k|_J = 0$ that guarantee no local contribution to the rates of change of the global constants of the motion. With the introduction of Lagrange multipliers ϑ_k^J and τ_J for the constraints, the $\dot{\gamma}_J^d$'s are found by solving the maximization problem

$$\max_{\dot{\gamma}_J^d} \, Y_J = \dot{s}|_J - \sum_k \vartheta_k^J \dot{c}_k|_J - \frac{k_\text{B} \tau_J}{2} \left(\dot{\gamma}_J^d \middle| \hat{G}_J \middle| \dot{\gamma}_J^d\right), \tag{49}$$

where the last constraint corresponds to the condition $(d\ell_J^d/dt)^2 = \text{const}$, necessary for maximizing with respect to direction only (see [69] for more details). Taking the variational derivative of Y_J with respect to $|\dot{\gamma}_J^d)$ and setting it equal to zero, we obtain

$$\frac{\delta Y_J}{|\delta \dot{\gamma}_J^d)} = \left|2(M)_\rho^J \gamma_J\right) - k_\text{B} \tau_J \, \hat{G}_J \left|\dot{\gamma}_J^d\right) = 0, \tag{50}$$

where we used the identity $(X| \hat{G}_J = \hat{G}_J |X)$, following from the symmetry of $\hat{G}_J$. We define (following [23,70]) the 'locally perceived non-equilibrium Massieu operator'

$$(M)_\rho^J = (S)_\rho^J - \sum_k \vartheta_k^J (C_k)_\rho^J . \tag{51}$$

Equation (50) yields

$$\left| \dot{\gamma}_J^d \right) = \frac{1}{k_B \tau_J} \hat{G}_J^{-1} \left| 2(M)_\rho^J \gamma_J \right) , \tag{52}$$

where the Lagrange multipliers ϑ_k^J (implicit in $(M)_\rho^J$) are the solution of the system of equations, obtained by substituting Equation (52) into the conservation constraints,

$$\left((C_\ell)_\rho^J \gamma_J \middle| \hat{G}_J^{-1} \middle| (M)_\rho^J \gamma_J \right) = 0 \quad \forall \ell . \tag{53}$$

This system can be solved explicitly for the ϑ_k^J's using Cramer's rule, to obtain convenient expressions for the $\dot{\gamma}_J^d$'s as ratios of determinants (as in the original formulations). The ϑ_k^J's are nonlinear functionals of ρ that may be interpreted as 'local non-equilibrium entropic potentials' conjugated with the conserved properties. For example, for $C_2 = H$, the Lagrange multiplier ϑ_2^J plays the role of 'local non-equilibrium inverse temperature' conjugated with the locally perceived energy, and for the stable equilibrium states of the SEA dynamics, it coincides with the thermodynamic inverse temperature $k_B \beta_J$ (see below).

Similarly to what was achieved in [69] for a non-composite system, we define the 'local non-equilibrium affinity' operators

$$\left| \Lambda_J \right) = \hat{G}_J^{-1/2} \left| 2(M)_\rho^J \gamma_J \right) , \tag{54}$$

so that the overall rate of entropy production becomes

$$\frac{ds(\rho)}{dt} = \sum_{J=1}^M \frac{(\Lambda_J | \Lambda_J)}{k_B \tau_J} . \tag{55}$$

$(\Lambda_J | \Lambda_J)$ is the norm of $2(M)_\rho^J \gamma_J$ with respect to the metric $\hat{G}_J^{-1}$ and may be interpreted as the 'degree of disequilibrium' of subsystem J. Hence, the necessary and sufficient condition for the overall state to be locally non-dissipative (no contribution to the overall entropy production from subsystem J) is that operator $2(M)_\rho^J \gamma_J$ vanishes.

However, in order for the equation of motion (41) to result independent of the unitary operators U used (in $\gamma_J = \sqrt{\rho_J} U$) to define the generalized square roots of ρ_J, we further restrict the choice of the metric superoperator $\hat{G}_J$. We assume that $\hat{G}_J = L_J^{-1} \hat{I}_J$, with L_J being some strictly positive, hermitian operator on $\mathcal{H}_J$, possibly a nonlinear function of ρ_J, so that $\hat{G}_J |X) = |L_J^{-1} X)$, $\hat{G}_J^{-1} |X) = |L_J X)$,

$$(X\gamma_J | \hat{G}_J^{-1} | Y\gamma_J) = \tfrac{1}{2} \mathrm{Tr} \left[\rho_J (X^\dagger L_J Y + Y^\dagger L_J X) \right] . \tag{56}$$

With the recollection of Equations (43) and (44), and using Equation (52), the dissipative term in Equation (41) becomes

$$-\{ \mathcal{D}_\rho^J, \rho_J \} = \frac{2}{k_B \tau_J} \left[L_J (M)_\rho^J \rho_J + \rho_J (M)_\rho^J L_J \right] , \tag{57}$$

and the system of equations that determines the Lagrange multipliers ϑ_k^J in $(M)_\rho^J$ is

$$\mathrm{Tr} \left(\rho_J \left[(C_\ell)_\rho^J L_J (M)_\rho^J + (M)_\rho^J L_J (C_\ell)_\rho^J \right] \right) = 0 \quad \forall \ell \tag{58}$$

so that the dependence on γ_J is only through the product $\gamma_J \gamma_J^\dagger$, i.e., the local state operator ρ_J.

The metric operator $L_J^{-1}/k_B \tau_J$ plays a role analogous to the symmetric thermal conductivity tensor $\hat{k}$ in heat transfer theory. In that context, $\hat{k}$ defines the general near-equilibrium linear relationship, $|q''\rangle = -\hat{k}|\nabla T\rangle$, between the heat flux vector q'' and the conjugated 'degree of disequilibrium' vector, i.e., the temperature gradient ∇T. Here, Equation (57) expresses a more general linear relationship between the local evolution operator $\dot{\gamma}_J^d$ and the non-equilibrium Massieu operator, $(S)_\rho^J - \sum_k \vartheta_k^J (C_k)_\rho^J$. This relation is more general, as it holds not only near equilibrium but also anywhere far from equilibrium. In the present quantum modeling context, it represents the nonlinear SEA extension into the far-non-equilibrium domain of Onsager's linear near-equilibrium theory, with reciprocity naturally embedded through the symmetry of any metric. For example, if L_J commutes with the local Hamiltonian H_J, its eigenvalues can assign different relaxation times to the local energy levels, capturing their uneven contributions to the rate of local energy redistribution described by the SEA dissipator $\mathcal{D}_\rho^J$.

Like in heat transfer, the conductivity tensor for an isotropic material is $\hat{k} = k\,\hat{I}$; here, an analogous simplification is obtained when $L_J = I_J$, the identity operator on $\mathcal{H}_J$. This corresponds to assuming a uniform Fisher–Rao metric.

Another noteworthy observation is that operators C_k, to be global constants of the motion (or 'charges'), must commute with the global Hamiltonian operator H, but they need not commute with each other. Therefore, the SEA formalism may also find application in the framework of quantum thermodynamic resource theories that contemplate 'non-commuting charges' and 'non-Abelian thermal states', as discussed in Yunger Halpern et al. [11] and Murthy et al. [71].

10. Simplest Composite-System SEA Equation of Motion

For simplicity, we proceed by assuming the following: (1) a uniform Fisher–Rao metric, with $\hat{G}_J = \hat{I}_J$, so that $\hat{G}_J|X\rangle = |X\rangle$ and $L_J = I_J$; and (2) only two global conserved properties, $c_1(\rho) = \mathrm{Tr}(\rho I)$ (the normalization condition) and $c_2(\rho) = \mathrm{Tr}(\rho H)$ (mean energy conservation).

The first assumption allows the obtaining of

$$\mathcal{D}_\rho^J = -\frac{2}{k_B \tau_J}\left[(S)_\rho^J - \sum_k \vartheta_k^J (C_k)_\rho^J\right]. \tag{59}$$

and the system of Equation (58) reduces to

$$\sum_k \vartheta_k^J \mathrm{Tr}\left[\rho_J\left\{(C_\ell)_\rho^J, (C_k)_\rho^J\right\}\right]$$
$$= \mathrm{Tr}\left[\rho_J\left\{(S)_\rho^J, (C_\ell)_\rho^J\right\}\right]. \tag{60}$$

The second assumption, $C_1 = I$ and $C_2 = H$, together with definitions (15) and (16), allows the writing of the SEA dissipators in the compact forms

$$-\{\mathcal{D}_\rho^J, \rho_J\} = \frac{2}{k_B \tau_J}\{\Delta(M)_\rho^J, \rho_J\}$$
$$= \frac{2}{k_B \tau_J}\frac{\begin{vmatrix} \{\Delta(S)_\rho^J, \rho_J\} & \{\Delta(H)_\rho^J, \rho_J\} \\ (H, S)_\rho^J & (H, H)_\rho^J \end{vmatrix}}{(H, H)_\rho^J}, \tag{61}$$

and the local non-equilibrium inverse temperatures conjugated with the locally perceived energy as

$$\vartheta_2^J = \frac{(H,S)_\rho^J}{(H,H)_\rho^J} \, . \tag{62}$$

As a result, the local evolution of each subsystem, J, is along the direction of steepest ascent of the locally perceived overall composite-system entropy, compatible with the conservation of the locally perceived overall composite-system energy and the conditions of unit-trace and non-negativity of the overall density operator, ρ. The rate of entropy production may be expressed as

$$\begin{aligned}
\frac{\mathrm{d}s(\rho)}{\mathrm{d}t} &= \sum_{J=1}^{M} \frac{4}{k_B \tau_J} (M,M)_\rho^J \\
&= \sum_{J=1}^{M} \frac{4}{k_B \tau_J} \frac{\begin{vmatrix} (S,S)_\rho^J & (H,S)_\rho^J \\ (H,S)_\rho^J & (H,H)_\rho^J \end{vmatrix}}{(H,H)_\rho^J} \, ,
\end{aligned} \tag{63}$$

showing clearly that it is non-negative since the numerators in the summation are Gram determinants. When the J-th term in the sum vanishes, we say that the local state ρ_J is 'non-dissipative'. If all local states are non-dissipative, then so is the overall state ρ. A non-dissipative ρ represents an equilibrium state if it commutes with H; otherwise, it belongs to a unitary limit cycle of the dynamics.

For ρ to be non-dissipative, there must exist β_Js, such that, for every J,

$$\rho_J \Delta (B\ln(\rho))_\rho^J = -\beta_J \rho_J \Delta(H)_\rho^J \, , \tag{64}$$

or, equivalently,

$$\rho_J \Delta \left[B\ln(\rho_J) + \beta_J H_J + \beta_J (V_{J\bar{J}})_\rho^J + (\mu_{J\bar{J}})_\rho^J \right] = 0 \, , \tag{65}$$

where $V_{J\bar{J}}$ is the Hamiltonian interaction operator defined in Equation (26), and $\mu_{J\bar{J}}$ the mutual information operator defined in Equation (27). Clearly, for $V_{J\bar{J}} = 0$ and $\mu_{J\bar{J}} = 0$, Equation (65) is satisfied by the Gibbs states

$$\rho_J = \exp\left(-\beta_J H_J \right) / \mathrm{Tr}\left[\exp\left(-\beta_J H_J \right) \right] \, . \tag{66}$$

Section 11.3 provides a numerical illustration of local time evolutions that converge to limit cycles obeying Equation (65). These examples also illustrate another noteworthy possibility. Specifically, without violating the no-signaling condition nor the second-law principle of global entropy non-decrease—which is always guaranteed by virtue of Equation (63) and the properties of Gram determinants—it is possible that the local entropy of a subsystem be decreasing during part of the time evolution. Indeed, for correlated states, the non-decrease in local entropy is not a second-law requirement. We hope that the observation that a local entropy decrease may be thermodynamically consistent in the presence of strong entanglement will stimulate the design of experimental verifications, as well as foundational discussions in the framework of quantum thermodynamic resource theories, even beyond and outside the SEA framework.

Regarding no-signaling, we note the following:

1. If subsystem J is non-interacting, $V_{J\bar{J}} = 0$, then

$$\Delta(H)_\rho^J = H_J - I_J \, \mathrm{Tr}\left(\rho_J H_J \right) = \Delta H_J \tag{67}$$

and

$$(H, H)_\rho^J = \mathrm{Tr}\left[\rho_J (\Delta H_J)^2\right] \tag{68}$$

depend only on the local H_J and ρ_J;

2. If J is uncorrelated, $\mu_{J\bar{J}} = 0$, then

$$\Delta(\mathrm{Bln}(\rho))_\rho^J = \mathrm{Bln}(\rho_J) - I_J \,\mathrm{Tr}\left(\rho_J \ln\left(\rho_J\right)\right) \tag{69}$$

and

$$(\mathrm{Bln}(\rho), \mathrm{Bln}(\rho))_\rho^J = \mathrm{Tr}\left[\rho_J \left(\ln\left(\rho_J\right)\right)^2\right] \tag{70}$$

depend only on the local ρ_J.

Therefore, it is only when J is both non-interacting and uncorrelated in that its local dissipation operator, $\mathcal{D}_\rho^J$, depends only on the local H_J and ρ_J. In this case, the local equation of motion Equation (41), along with $\mathcal{D}_\rho^J$ given by Equation (61), reduces exactly to the non-composite system version of SEA evolution [16]. Instead, if J is either interacting or correlated, $\mathcal{D}_\rho^J$ and, therefore, the local nonlinear SEA evolution, according to Equations (41) and (61), is determined not only by the local H_J and ρ_J but also by the local perceptions of the overall Hamiltonian operator H and/or the overall entropy operator $\mathrm{Bln}(\rho)$, nonetheless without violating the no-signaling condition.

To prove no-signaling, assume that subsystem J is correlated but not interacting with any of the subsystems in $\bar{J}$. Now, switch on an arbitrary interaction that may involve the subsystems in $\bar{J}$ but not subsystem J, so that the only change in the overall Hamiltonian H is the term $H_{\bar{J}}$, which changes to $H'_{\bar{J}}$. Within J, the locally perceived deviation operators are $\Delta(H)_\rho^J$ and $\Delta(\mathrm{Bln}(\rho))_\rho^J$, and hence, the SEA dissipator $\mathcal{D}_\rho^J$, as well as all the terms in the RHS of Equation (41), are not modified by such a change. Therefore, acting within $\bar{J}$ makes it impossible to affect the time evolution of ρ_J and any local observable of J.

We also mention the following in the spirit of non-linear evolutions. In the case of a general open quantum dynamics, the requirement that a map, $\mathcal{W}_t(\rho)$, be completely positive and trace-preserving (CPTP) is quite restrictive [72–74]. In fact, the reduced dynamics of the subsystem in interaction with the environment need not be CPTP [75–77]. Here, we define CPTP as follows. If $\mathcal{M}(\rho_s)$ is a map acting on a subsystem in state ρ_s, which interacts with an initially uncorrelated environment in state ρ_E and is positive and trace-preserving—meaning it preserves the trace and ensures that the evolved matrix remains semi-definite (maintaining probability conservation)—then $\mathcal{M}(\rho_s)$ is CPTP if and only if the extended map, $\Lambda : \mathcal{M}(\rho_s) \otimes I_N$, remains positive for all Ns [72]. As has been argued in the literature, the CPTP condition is restrictive because, along with the underlying assumption of Markovianity, it requires a preparation of the initial state into a product state of system and environment. But, if there are strong initial correlations between system and environment, or in fact, if the evolution is non-Markovian in nature, then this requirement fails, and we are forced to consider PTP (positive and trace-preserving) maps only [78,79]. Given that we are dealing with a theory that does not restrict itself to Markovian evolution, and that there is no imposition of weak interaction between the subsystems, SEA being PTP satisfies the requirements for a suitable nonlinear model of quantum thermodynamics prohibiting signaling.

11. Non-Interacting Qubits

In extreme cases, Equation (65) shows that, even if the subsystems are entangled, and therefore, the local states ρ_J are mixed, operators $\mathcal{D}_\rho^J$ may vanish. Equations (39) and (41) reduce to the standard Schrödinger equation, and the trajectory in state space is a limit cycle of the SEA dynamics. A noteworthy particular case is when the overall system is in a pure state. Then, $\text{Bln}(\rho) = 0$ and standard unitary evolutions of pure states emerge as limit cycles of the nonlinear SEA dynamics. In this section, we discuss a few less trivial examples, to numerically illustrate some general features of Equation (39) with $\mathcal{D}_\rho^J$ given by Equation (61).

We consider examples of mixed and correlated initial states of a two-qubit composite, AB, that belong to the special class

$$
\begin{aligned}
\rho &= \tfrac{1}{4}\left[\mathrm{I}_4 + \sum_{j=\{x,y,z\}} (a_j\,\sigma_j{\otimes}\mathrm{I}_2 + b_j\,\mathrm{I}_2{\otimes}\sigma_j + c_j\,\sigma_j{\otimes}\sigma_j)\right]\\
&= \tfrac{1}{4}\begin{pmatrix}
1+a_z+b_z+c_z & b_x-\mathrm{i}b_y & a_x-\mathrm{i}a_y & c_x-c_y\\
b_x+\mathrm{i}b_y & 1+a_z-b_z-c_z & c_x+c_y & a_x-\mathrm{i}a_y\\
a_x+\mathrm{i}a_y & c_x+c_y & 1-a_z+b_z-c_z & b_x-\mathrm{i}b_y\\
c_x-c_y & a_x+\mathrm{i}a_y & b_x+\mathrm{i}b_y & 1-a_z-b_z+c_z
\end{pmatrix},
\end{aligned}
\tag{71}
$$

with real a_j's, b_j's, c_j's, such that $a^2 = \boldsymbol{a}{\cdot}\boldsymbol{a} \leq 1$, $b^2 = \boldsymbol{b}{\cdot}\boldsymbol{b} \leq 1$, $c^2 = \boldsymbol{c}{\cdot}\boldsymbol{c} \leq 3 - a^2 - b^2$, plus other conditions necessary for non-negativity (see e.g., [80–82]). We will denote the eigenvalues of ρ as λ_j and, for shorthand, define

$$
\eta_j = -k_{\mathrm{B}}\text{Bln}(\lambda_j)\,.
\tag{72}
$$

We further assume that the two qubits are non-interacting and have local Hamiltonian operators given by $H_J = \omega_J \boldsymbol{h}^J \cdot \boldsymbol{\sigma}_J$, where $\boldsymbol{\sigma}_J$ denotes the 3-vector formed by the local Pauli operators of subsystem J, and $\boldsymbol{h}^J$ is the local Hamiltonian unit 3-vector, so that

$$
\begin{aligned}
H &= H_A{\otimes}\mathrm{I}_2 + \mathrm{I}_2{\otimes}H_B\\
&= \omega_A\boldsymbol{h}^A{\cdot}\boldsymbol{\sigma}_A{\otimes}\mathrm{I}_2 + \mathrm{I}_2{\otimes}\omega_B\boldsymbol{h}^B{\cdot}\boldsymbol{\sigma}_B\,,\\
(H)_\rho^A &= H_A + \mathrm{I}_2\,\omega_B\boldsymbol{h}^B{\cdot}\boldsymbol{b}\,,\\
(H)_\rho^B &= \mathrm{I}_2\,\omega_A\boldsymbol{h}^A{\cdot}\boldsymbol{a} + H_B\\
\Delta(H)^A &= \Delta H_A\,,\quad \Delta(H)^B = \Delta H_B\,,\\
(H,H)_\rho^A &= [1-(\boldsymbol{h}^A{\cdot}\boldsymbol{a})^2]\,\omega_A^2\,,\\
(H,H)_\rho^B &= [1-(\boldsymbol{h}^B{\cdot}\boldsymbol{b})^2]\,\omega_B^2\,,
\end{aligned}
\tag{73}
$$

11.1. Bell Diagonal States

Equation (71) gives Bell diagonal states [83,84] (BDS) if $a_j = b_j = 0$ for all js (and Werner states [85] if, in addition, $c_j = 4w/3 - 1$ for all js),

$$
\begin{aligned}
\rho^{\text{Bell}} &= \tfrac{1}{4}\begin{pmatrix}
1+c_z & 0 & 0 & c_x-c_y\\
0 & 1-c_z & c_x+c_y & 0\\
0 & c_x+c_y & 1-c_z & 0\\
c_x-c_y & 0 & 0 & 1+c_z
\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}
\lambda_2+\lambda_3 & 0 & 0 & \lambda_3-\lambda_2\\
0 & \lambda_1+\lambda_4 & \lambda_4-\lambda_1 & 0\\
0 & \lambda_4-\lambda_1 & \lambda_1+\lambda_4 & 0\\
\lambda_3-\lambda_2 & 0 & 0 & \lambda_2+\lambda_3
\end{pmatrix},\\
&= \rho_A{\otimes}\rho_B + \sum_{j=\{x,y,z\}} c_j\,\sigma_j{\otimes}\sigma_j
\end{aligned}
\tag{74}
$$

whose eigen-decomposition may be written as

$$\rho^{\text{Bell}} = U^{\text{Bell}}\text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)(U^{\text{Bell}})^\dagger,$$

$$U^{\text{Bell}} = \frac{\sigma_x \otimes I_2 - \sigma_z \otimes \sigma_x}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$$\lambda_1 = \tfrac{1}{4}(1 - c_x - c_y - c_z), \quad \lambda_2 = \tfrac{1}{4}(1 - c_x + c_y + c_z),$$
$$\lambda_3 = \tfrac{1}{4}(1 + c_x - c_y + c_z), \quad \lambda_4 = \tfrac{1}{4}(1 + c_x + c_y - c_z). \tag{75}$$

The overall entropy operator, Equation (18), becomes

$$S(\rho^{\text{Bell}}) = U^{\text{Bell}}[\text{diag}(\eta_1, \eta_2, \eta_3, \eta_4)](U^{\text{Bell}})^\dagger$$

$$= \frac{1}{2}\begin{pmatrix} \eta_2 + \eta_3 & 0 & 0 & \eta_3 - \eta_2 \\ 0 & \eta_1 + \eta_4 & \eta_4 - \eta_1 & 0 \\ 0 & \eta_4 - \eta_1 & \eta_1 + \eta_4 & 0 \\ \eta_3 - \eta_2 & 0 & 0 & \eta_2 + \eta_3 \end{pmatrix}, \tag{76}$$

and the local operators of the SEA formalism

$$\rho_A = \rho_B = \tfrac{1}{2}I_2,$$
$$(\text{Bln}(\rho))^A = (\text{Bln}(\rho))^B = -\tfrac{q}{4k_\text{B}}I_2,$$
$$\{(\text{Bln}(\rho))^A, \rho_A\} = \{(\text{Bln}(\rho))^B, \rho_B\} = -\tfrac{q}{4k_\text{B}}I_2,$$
$$\{\Delta(\text{Bln}(\rho))^A, \rho_A\} = \{\Delta(\text{Bln}(\rho))^B, \rho_B\} = 0,$$
$$(H, H)^A_\rho = \omega_A^2, \quad (H, H)^B_\rho = \omega_B^2,$$
$$(H, \text{Bln}(\rho))^A_\rho = (H, \text{Bln}(\rho))^B_\rho = 0. \tag{77}$$

where q is defined as

$$q = \eta_1 + \eta_2 + \eta_3 + \eta_4. \tag{78}$$

Therefore, somewhat surprisingly, we find that

$$-\{\mathcal{D}^A_\rho, \rho_A\} = -\{\mathcal{D}^B_\rho, \rho_B\} = 0, \tag{79}$$

i.e., the Bell diagonal states are non-dissipative limit cycles of the nonlinear SEA dynamics under any Hamiltonian. But most neighboring and other states in the class defined by Equation (71) are dissipative, as we see in the following examples.

11.2. Separable, but Correlated Mixed States

For a simple example of correlated but separable mixed states, assume Equation (71) with $a_x = a$, $b_z = b$, and $a_y = a_z = b_x = b_y = c_x = c_y = c_z = 0$, so that

$$\rho = \tfrac{1}{4}\begin{pmatrix} 1+b & 0 & a & 0 \\ 0 & 1-b & 0 & a \\ a & 0 & 1+b & 0 \\ 0 & a & 0 & 1-b \end{pmatrix} = \rho_A \otimes \rho_B - \tfrac{1}{4}ab\,\sigma_x \otimes \sigma_z,$$

$$\lambda_1 = \tfrac{1}{4}(1 - a - b), \quad \lambda_2 = \tfrac{1}{4}(1 - a + b),$$
$$\lambda_3 = \tfrac{1}{4}(1 + a - b), \quad \lambda_4 = \tfrac{1}{4}(1 + a + b). \tag{80}$$

If the two non-interacting qubits have local Hamiltonians $H_A = \sigma_z$ and $H_B = \sigma_x$, we find

$$-\{\mathcal{D}_\rho^A, \rho_A\} = \frac{(1-a^2)(bf-g)}{2k_B\tau_A}\sigma_x, \tag{81}$$

$$-\{\mathcal{D}_\rho^B, \rho_B\} = \frac{(1-b^2)(af-p)}{2k_B\tau_B}\sigma_z, \tag{82}$$

where f, g, and p are defined as

$$\begin{aligned}
f &= \eta_1 - \eta_2 - \eta_3 + \eta_4, \\
g &= \eta_1 + \eta_2 - \eta_3 - \eta_4, \\
p &= \eta_1 - \eta_2 + \eta_3 - \eta_4,
\end{aligned} \tag{83}$$

so that the nonlinear evolution is clearly nontrivial. However, it preserves the zero mean energies of both qubits, and while the overall entropy increases and mutual information partially fades away, it drives the overall state towards a non-dissipative correlated state with maximally mixed marginals. We proved above that signaling is impossible, even though $\mathcal{D}_\rho^A$ depends not only on a but also on b, and $\mathcal{D}_\rho^B$ on a, which agrees with our no-signaling condition in Equation (2).

Figure 2 shows a Bloch-ball representation of the time evolutions of the local density operators ρ_A and ρ_B for an initial state in the class considered in this section, Equation (80), with $a = -0.6$ and $b = 0.4$. The time evolution is computed through a numerical integration of Equation (39). The local evolutions of the two non-interacting qubits approach asymptotically the respective local Gibbs states, i.e., the states of maximum local entropy for the given initial mean local energies, for which Equation (65) reduce to

$$\begin{aligned}
\Delta(\text{Bln}(\rho_A)) &= -\beta_A\,\Delta H_A, \\
\Delta(\text{Bln}(\rho_B)) &= -\beta_B\,\Delta H_B.
\end{aligned} \tag{84}$$

Mutual information, $S(\rho_A) + S(\rho_B) - S(\rho)$, and a measure of coherence, $\text{Tr}(\rho^2 H^2 - \rho H\rho H) = \frac{1}{2}\text{Tr}(K^\dagger K)$, where $K = i[H,\rho]$, are reduced through the SEA dissipation terms but do not vanish. Indeed, the Bell observable, $\text{Tr}(\rho\,\sigma_x\otimes\sigma_z)$, which is initially zero, builds up to a periodic oscillation of constant amplitude.

11.3. Entangled, Separable, and Correlated Mixed States

For a slightly more elaborate example that includes entangled mixed states, assume Equation (71) with $a_x = a_z = a/\sqrt{2}$, $b_x = b_z = b/\sqrt{2}$, and $c_x = c_y = c_z = 2(a-b)/3$, so that the eigenvalues of ρ and those of its partial transpose are

$$\begin{aligned}
\lambda_1 &= \tfrac{1}{4}(1+a-b), & \lambda_2 &= \tfrac{1}{12}(3-a-5b), \\
\lambda_3 &= \tfrac{1}{12}(3+5a+b), & \lambda_4 &= \tfrac{1}{12}(3-7a+7b),
\end{aligned} \tag{85}$$

and

$$\begin{aligned}
\lambda_1^{PT} &= \tfrac{1}{12}(3+a-b), & \lambda_2^{PT} &= \tfrac{1}{12}(3-5a+5b), \\
\lambda_3^{PT} &= \tfrac{1}{12}(3+2a-2b+\sqrt{d}), & & \\
\lambda_4^{PT} &= \tfrac{1}{12}(3+2a-2b-\sqrt{d}), & &
\end{aligned} \tag{86}$$

with $d = 25a^2 - 14ab + 25b^2$. Figure 3 shows the complete set of admissible pairs of values of parameters a and b for the set of correlated mixed states considered in this example, which encompasses both separable and entangled states. For instance, considering $a = -b$

($\lambda_2 = \lambda_3$), these states are separable for $-3/14 \le b \le 1/4$ and entangled for $1/4 < b \le 1/2$. We compute explicitly each term in Equation (61), also for this class of states, to find

$$\{(S)^A, \rho_A\} = -\frac{(1-a^2)(f-5bp)}{20\sqrt{2}}(\sigma_x+\sigma_z),$$

$$\{(S)^B, \rho_B\} = \frac{(1-b^2)(g+5ap)}{20\sqrt{2}}(\sigma_x+\sigma_z),$$

(87)

$$(H,S)^A_\rho = -\frac{(1-a^2)(f-5bp)\omega_A}{20\sqrt{2}}(h^A_x+h^A_z),$$

$$(H,S)^B_\rho = \frac{(1-b^2)(g+5ap)\omega_B}{20\sqrt{2}}(h^B_x+h^B_z),$$

(88)

$$(H,H)^A_\rho = [1-\tfrac{1}{2}(h^A_x+h^A_z)^2 a^2]\omega^2_A,$$

$$(H,H)^B_\rho = [1-\tfrac{1}{2}(h^B_x+h^B_z)^2 b^2]\omega^2_B,$$

(89)

$$-\{\mathcal{D}^A_\rho, \rho_A\} = -\frac{(1-a^2)(f-5bp)}{5\sqrt{2}[2-(h^A_x+h^A_z)^2 a^2]k_B\tau_A}\zeta_A,$$

$$-\{\mathcal{D}^B_\rho, \rho_B\} = \frac{(1-b^2)(g+5ap)}{5\sqrt{2}[2-(h^B_x+h^B_z)^2 b^2]k_B\tau_B}\zeta_B.$$

(90)

Here, f, g, and p are defined as

$$f = 3\eta_1 - 5\eta_2 + 5\eta_3 - 3\eta_4,$$

$$g = 3\eta_1 + 5\eta_2 - 5\eta_3 - 3\eta_4,$$

$$p = \eta_1 - \eta_2 - \eta_3 + \eta_4,$$

(91)

$$\zeta_A = \left[1 - \tfrac{1}{2}(h^A_x+h^A_z)^2 a^2 + \tfrac{1}{2}(h^A_x+h^A_z)a^2\right](\sigma_x+\sigma_z)$$
$$- (h^A_x+h^A_z)\boldsymbol{h}^A \cdot \boldsymbol{\sigma}_A,$$

$$\zeta_B = \left[1 - \tfrac{1}{2}(h^B_x+h^B_z)^2 b^2 + \tfrac{1}{2}(h^B_x+h^B_z)b^2\right](\sigma_x+\sigma_z)$$
$$- (h^B_x+h^B_z)\boldsymbol{h}^B \cdot \boldsymbol{\sigma_B}.$$

(92)

For example, if the two non-interacting qubits A and B have local Hamiltonians $H_A = \sigma_z$ and $H_B = \sigma_x$, we find

$$-\{\mathcal{D}^A_\rho, \rho_A\} = -\frac{(1-a^2)(f-5bp)}{5\sqrt{2}(2-a^2)k_B\tau_A}\sigma_x,$$

(93)

$$-\{\mathcal{D}^B_\rho, \rho_B\} = \frac{(1-b^2)(g+5ap)}{5\sqrt{2}(2-b^2)k_B\tau_B}\sigma_z,$$

(94)

so that again the nonlinear evolution is clearly nontrivial in the sense that the local nonlinear evolution of A (B) does not depend only on ρ_A (ρ_B), despite being no-signaling.

For initial states in the class considered in this section, with parameter values a and b corresponding to the four points in Figure 3, Figures 4–7 depict the typical time evolution of the local density operators ρ_A and ρ_B in the local Bloch-balls. These results are obtained from the numerical integration of the steepest locally perceived entropy ascent equation of motion, Equation (39), for non-interacting subsystems A and B.
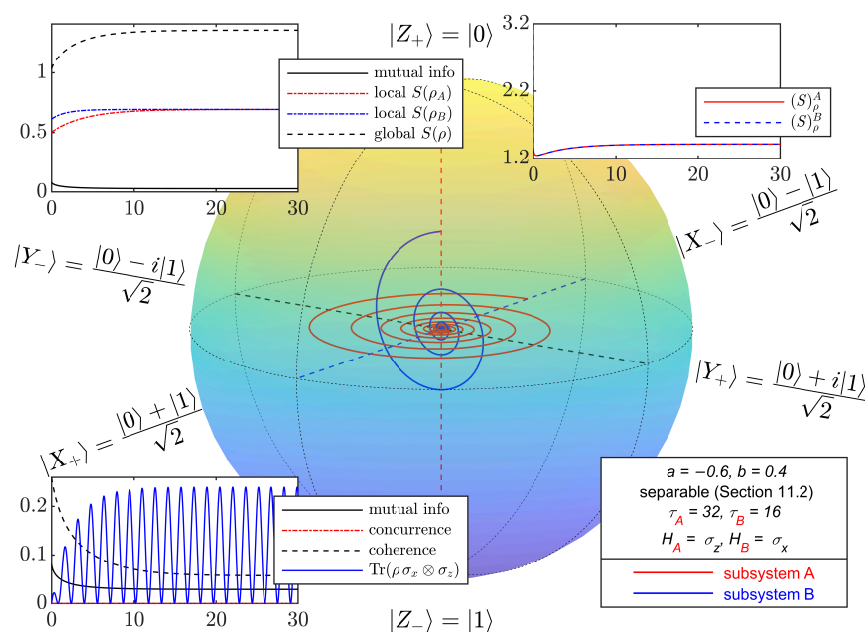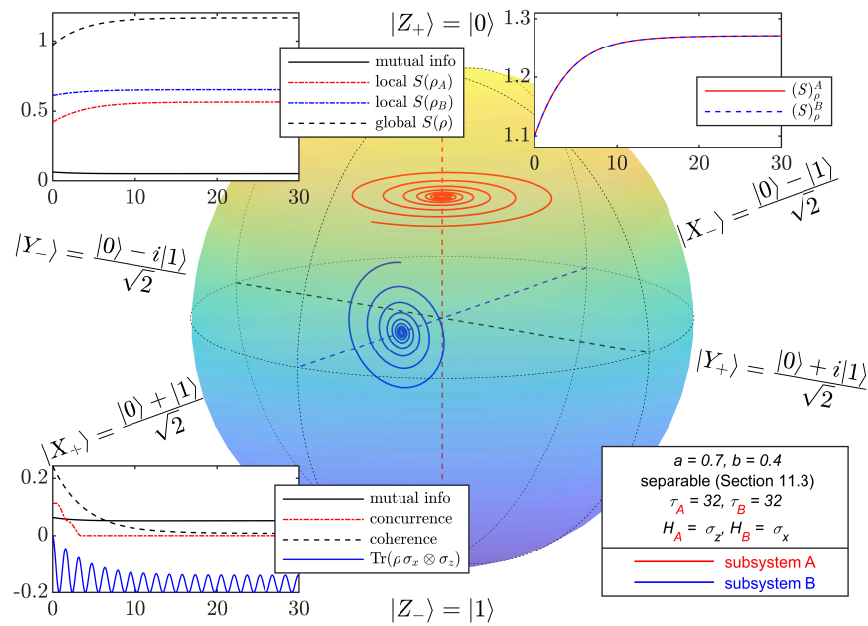
Figure 2. Bloch-ball representation of the time evolutions of the local density operators ρ_A and ρ_B of non-interacting qubits A and B with local Hamiltonians $H_A = \sigma_z$ and $H_B = \sigma_x$. The evolution follows the steepest locally perceived entropy ascent equation, Equation (39), with $\hbar = 1$ and $k_B = 1$. The initial state of the composite system AB is the separable, correlated mixed state given by Equation (80) with $a = -0.6$ and $b = 0.4$. Insets show the time evolutions of global $[S(\rho)]$, local $[S(\rho_A), S(\rho_B)]$, and locally perceived $[(S)_\rho^A = (S)_\rho^B]$ entropies, mutual information $[S(\rho_A) + S(\rho_B) - S(\rho)]$, concurrence—a coherence measure $[\mathrm{Tr}(\rho^2 H^2 - \rho H \rho H)]$, and the Bell observable $\mathrm{Tr}(\rho\,\sigma_x \otimes \sigma_z)$. The local evolutions approach the respective local Gibbs states. Without violating the no-signaling condition, dissipation causes a (non-complete) reduction in mutual information and coherence, while the Bell observable builds up to a steady state oscillation.
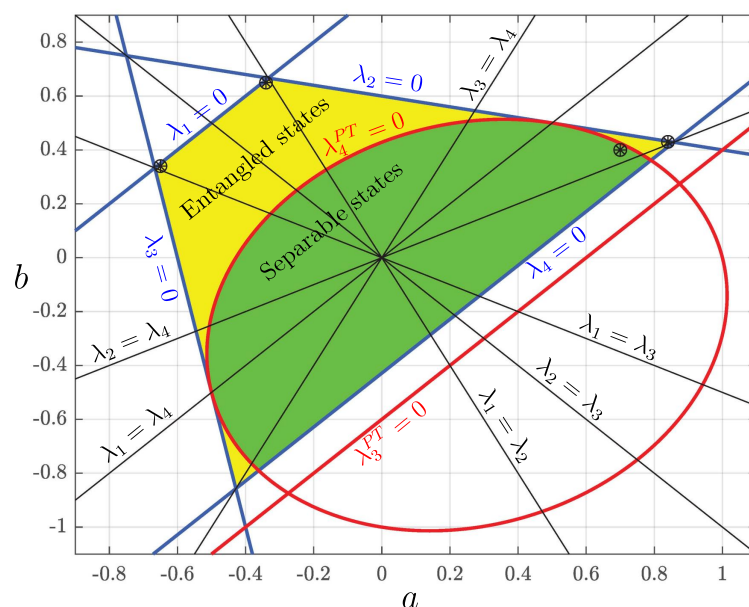
**Figure 3.** Ranges of admissible values of parameters a and b for the separable and entangled initial states in the class defined in Section 11.3. The four points denoted with the symbol ⊛ represent the initial states chosen to illustrate the local time evolutions in Figures 4–7. The two points near the $\lambda_1 = 0$ line represent the strongly entangled states for which Figures 6 and 7 show a local entropy decrease for one of the subsystems, while the second-law principle of global entropy non-decrease is not violated.

Figure 4. Bloch-ball representation of the time evolutions of the local density operators ρ_A and ρ_B of non-interacting qubits A and B with local Hamiltonians $H_A = \sigma_z$ and $H_B = \sigma_x$, resulting from numerical integration of the steepest locally perceived entropy ascent equation of motion, Equation (39). The initial state of the composite system AB is the separable, correlated mixed state in the class of states defined in Section 11.3 with $a = 0.7$ and $b = 0.4$. The local evolutions approach the respective local Gibbs states. Without violating the no-signaling condition, dissipation causes a (non-complete) reduction in mutual information and coherence, while the Bell observable converges to a steady state oscillation. Insets are similar to Figure 2.
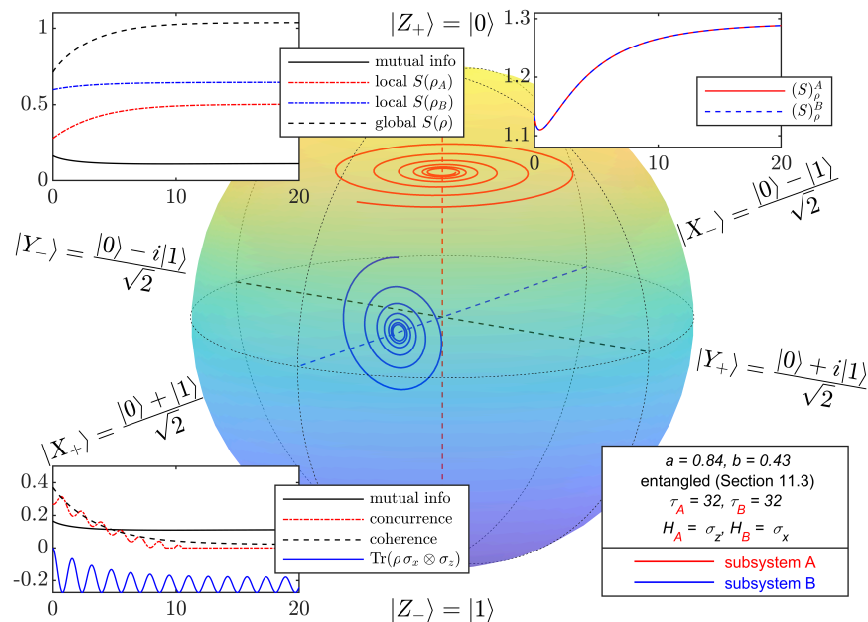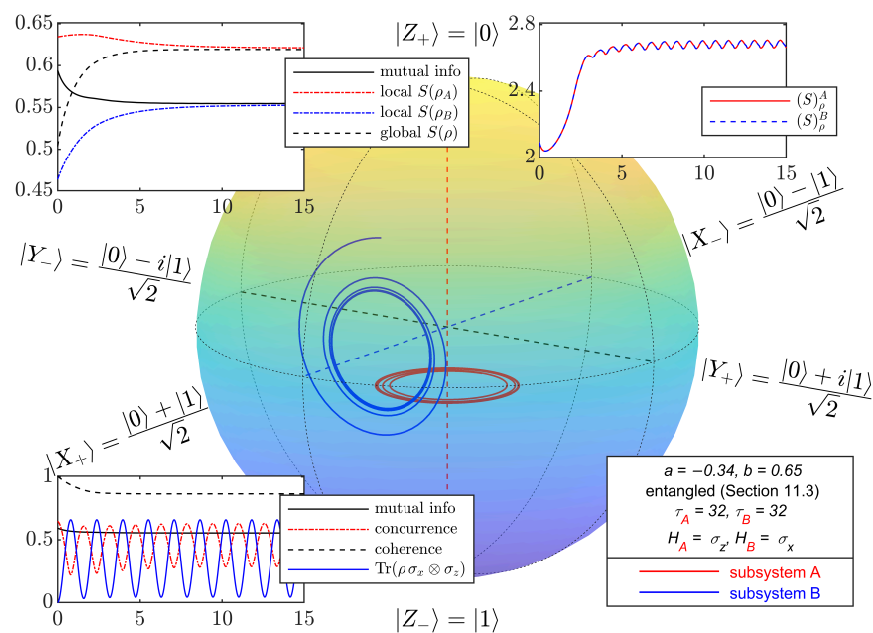

Figure 5. Bloch-ball representations of the time evolution of two non-interacting qubits, obtained from numerical integration of Equation (39) with initial entangled mixed states from the class defined in Section 11.3. The values of (a, b) are $(0.84, 0.43)$. Due to entanglement, local evolutions approach limit cycles with local entropy lower than the corresponding local Gibbs states. Insets are similar to Figure 2.
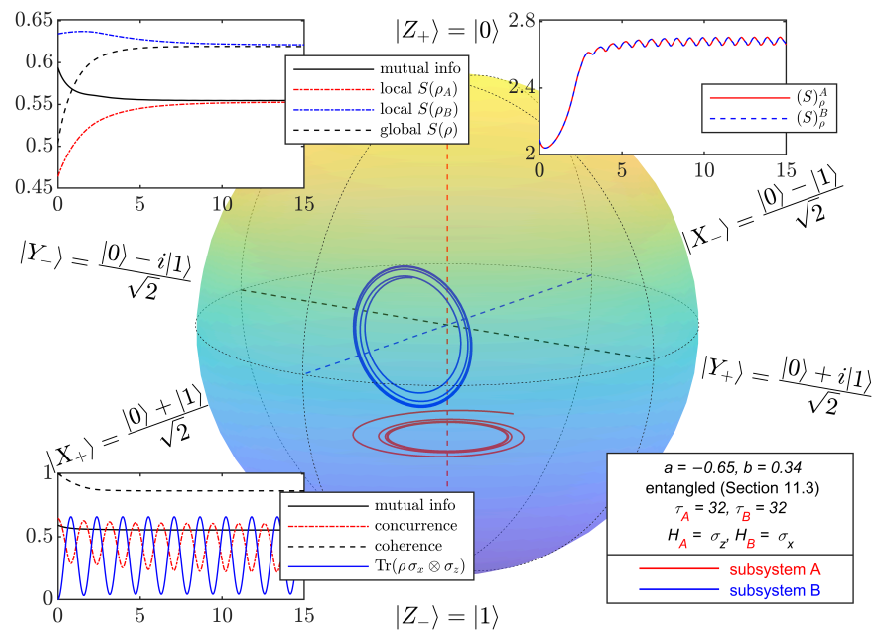
Figure 6. Bloch-ball representations of the time evolution of two non-interacting qubits, obtained from numerical integration of Equation (39) with initial entangled mixed states from the class defined in Section 11.3. The values of (a, b) are $(-0.65, 0.34)$. Due to entanglement, local evolutions approach limit cycles with local entropy significantly lower than the corresponding local Gibbs states. Strong entanglement leads to a local entropy decrease in subsystem *B* and *A*, respectively, without violating the global entropy non-decrease dictated by the second law. Insets are presented in Figure 2.

Figure 7. Bloch-ball representations of the time evolution of two non-interacting qubits, obtained from numerical integration of Equation (39) with initial entangled mixed states from the class defined in Section 11.3. The values of (a, b) are $(-0.34, 0.65)$. Due to entanglement, local evolutions approach limit cycles with local entropy significantly lower than the corresponding local Gibbs states. Strong entanglement leads to a local entropy decrease in subsystem *B* and *A*, respectively, without violating the global entropy non-decrease dictated by the second law. Insets are similar to Figure 2.

Whereas, for a separable initial state, Figure 4 shows that the local states approach the respective local Gibbs states, it is clear from Figures 5–7 that the presence of entanglement does not allow the local states to approach the respective local Gibbs states. Indeed, the

local evolutions approach non-dissipative unitary limit cycles where $\{\mathcal{D}^A, \rho_A\} = 0$ and $\{\mathcal{D}^B, \rho_B\} = 0$, and therefore, Equation (65) is satisfied. Without violating the no-signaling condition, dissipation causes a (non-complete) reduction in mutual information and coherence, while the Bell observable $\text{Tr}(\rho\, \sigma_x \otimes \sigma_z)$ and concurrence converge to steady state oscillations.

12. Conclusions

In this work, we have explored the no-signaling principle and its implications for nonlinear dynamics in quantum theory. We examined the measure-theoretic representation of mixed ensembles by providing a consistent framework for handling nonlinearities and describe ensemble mixing unambiguously. This approach also offers a conceptual basis for developing nonlinear quantum dynamical models for non-equilibrium systems and addressing unresolved questions about individual quantum states. Additionally, we reviewed the original philosophical motivations underlying the steepest entropy ascent (SEA) nonlinear evolution law, which trades linearity for strong thermodynamic consistency.

Our analysis demonstrates that the SEA formalism provides a valid framework for describing evolution or designing master equations. These can be applied in nonlinear extensions of quantum mechanics (QM) or in phenomenological models of open quantum systems within quantum thermodynamics. Our key focus was the no-signaling principle. We introduced a criterion for no-signaling based on local perception operators (LPOs), a generalization of the traditional density-operator-based definition. Unlike the conventional assumption that local evolution depends solely on local properties, SEA formalism incorporates the local effects of preexisting correlations and coherence through LPOs. Our detailed examination of the foundations and properties of LPOs highlights their potential as key tools for constructing signaling-free, nonlinear, and non-local non-equilibrium dynamical models, even outside of the SEA framework.

By inherently respecting the no-signaling condition, the SEA formalism upholds the principle of causality and aligns with the axiomatic paradigm proposed by Popescu and Rohrlich [86] for quantum theories. Furthermore, the SEA approach addresses conceptual challenges such as the Schrödinger–Park paradox, offering insights into a thermodynamically consistent integration of quantum mechanics and non-equilibrium thermodynamics. Related to the no-signaling property, our discussion of LPOs and supporting numerical examples strengthens the conjecture—supported by prior numerical results from Cano-Andrade et al. [25]—that SEA evolution is universally compatible with decoherence principles. Specifically, SEA evolution ensures that correlations and entanglement cannot build up between non-interacting systems but can decay partially or completely, a process mediated via LPOs.

An intriguing implication of SEA formalism is its compatibility with thermodynamic principles while allowing for scenarios where, under strong entanglement, the local entropy of a subsystem may temporarily decrease during the evolution, without violating the no-signaling condition or the global entropy non-decrease dictated by the second law. We emphasize that, differently from the Lindbladian 'bottom-up' approach in which quantum thermodynamic second law(s) emerge from statistical approximations of the linear unitary dynamics induced by system-environment interactions, the SEA 'top-down' approach builds strong second-law compatibility directly into the nonequilibrium dynamics variational principle, constrained by the phenomenological details about the system's structure, subsystems' interactions, and environmental effects. However, because of the inherent nonlinearity of the SEA formalism, the solution becomes more complex, and a full analytical calculation can become intractable [27]. Additionally, the relaxation times τ_J remain

heuristic parameters, which are not derived from first principles and need to be tuned with respect to a given experimental set-up.

We hope this study provides a foundational basis for exploring applications of SEA modeling in quantum computing. Our comprehensive approach and observations aim to stimulate constructive discussions and inspire further studies, particularly in developing thermodynamically consistent models for non-equilibrium processes. These findings have potential applications in modeling the dynamics of quantum systems, advancing quantum technologies, and contributing to the framework of quantum thermodynamic resource theories, even beyond the SEA paradigm.

Author Contributions: R.K.R.: idea. R.K.R. and G.P.B.: contributed equally to the computation and writing of the manuscript. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. Kosloff, R. Quantum Thermodynamics and Open-Systems Modeling. *J. Chem. Phys.* **2019**, *150*, 204105. [CrossRef]
2. Chruściński, D. Dynamical Maps beyond Markovian Regime. *Phys. Rep.* **2022**, *992*, 1–85. [CrossRef]
3. Schmidt, H.J.; Gemmer, J. Stochastic Thermodynamics of a Finite Quantum System Coupled to Two Heat Baths. *Entropy* **2023**, *25*, 504. [CrossRef]
4. Riera-Campeny, A.; Sanpera, A.; Strasberg, P. Quantum Systems Correlated with a Finite Bath: Nonequilibrium Dynamics and Thermodynamics. *PRX Quantum* **2021**, *2*, 010340. [CrossRef]
5. Manzano, G.; Subero, D.; Maillet, O.; Fazio, R.; Pekola, J.P.; Roldán, É. Thermodynamics of Gambling Demons. *Phys. Rev. Lett.* **2021**, *126*, 080603. [CrossRef] [PubMed]
6. Simmons, R.F.; Park, J.L. The Essential Nonlinearity of N-level Quantum Thermodynamics. *Found. Phys.* **1981**, *11*, 297–305. [CrossRef]
7. Beretta, G.P. A Theorem on Lyapunov Stability for Dynamical Systems and a Conjecture on a Property of Entropy. *J. Math. Phys.* **1986**, *27*, 305–308. [CrossRef]
8. Polkovnikov, M.; Gramolin, A.V.; Kaplan, D.E.; Rajendran, S.; Sushkov, A.O. Experimental Limit on Nonlinear State-Dependent Terms in Quantum Theory. *Phys. Rev. Lett.* **2023**, *130*, 040202. [CrossRef]
9. Melnychuk, O.; Giaccone, B.; Bornman, N.; Cervantes, R.; Grassellino, A.; Harnik, R.; Kaplan, D.E.; Nahal, G.; Pilipenko, R.; Posen, S.; et al. An Improved Bound on Nonlinear Quantum Mechanics Using a Cryogenic Radio Frequency Experiment. *arXiv* **2024**, arXiv:quant-ph/2411.09611. [CrossRef]
10. Somhorst, F.H.B.; van der Meer, R.; Correa Anguita, M.; Schadow, R.; Snijders, H.J.; de Goede, M.; Kassenberg, B.; Venderbosch, P.; Taballione, C.; Epping, J.P.; et al. Quantum Simulation of Thermodynamics in an Integrated Quantum Photonic Processor. *Nat. Commun.* **2023**, *14*, 3895. [CrossRef]
11. Yunger Halpern, N.; Faist, P.; Oppenheim, J.; Winter, A. Microcanonical and Resource-Theoretic Derivations of the Thermal State of a Quantum System with Noncommuting Charges. *Nat. Commun.* **2016**, *7*, 12051. [CrossRef]
12. Hatsopoulos, G.N.; Gyftopoulos, E.P. A Unified Quantum Theory of Mechanics and Thermodynamics. Part I. Postulates. *Found. Phys.* **1976**, *6*, 15–31. [CrossRef]

13. Hatsopoulos, G.N.; Gyftopoulos, E.P. A Unified Quantum Theory of Mechanics and Thermodynamics. Part IIa. Available Energy. *Found. Phys.* **1976**, *6*, 127–141. [CrossRef]

14. Hatsopoulos, G.N.; Gyftopoulos, E.P. A Unified Quantum Theory of Mechanics and Thermodynamics. Part IIb. Stable Equilibrium States. *Found. Phys.* **1976**, *6*, 439–455. [CrossRef]

15. Hatsopoulos, G.N.; Gyftopoulos, E.P. A Unified Quantum Theory of Mechanics and Thermodynamics. Part III. Irreducible Quantal Dispersions. *Found. Phys.* **1976**, *6*, 561–570. [CrossRef]

16. Beretta, G.P.; Gyftopoulos, E.P.; Park, J.L.; Hatsopoulos, G.N. Quantum Thermodynamics. A New Equation of Motion for a Single Constituent of Matter. *Il Nuovo Cimento B* **1984**, *82*, 169–191. [CrossRef]

17. Beretta, G.P.; Gyftopoulos, E.P.; Park, J.L. Quantum Thermodynamics. A New Equation of Motion for a General Quantum System. *Il Nuovo Cimento B* **1985**, *87*, 77–97. [CrossRef]

18. Maddox, J. Uniting Mechanics and Statistics. *Nature* **1985**, *316*, 11. [CrossRef]

19. Beretta, G.P. The Role of Stability in the Unification of Mechanics and Thermodynamics. In *Colloque International sur la Stabilité: Recueil des Conférences*; Pélegrin, M., Ed.; ANAE-ONERA-CERT: Toulouse, France, 1987; pp. 87–104. Available online: https://hal.science/hal-04050717 (accessed on 19 September 2025).

20. Gheorghiu-Svirschevski, S. Nonlinear Quantum Evolution with Maximal Entropy Production. *Phys. Rev. A* **2001**, *63*, 022105. [CrossRef]

21. Gheorghiu-Svirschevski, S. Addendum to "Nonlinear Quantum Evolution with Maximal Entropy Production". *Phys. Rev. A* **2001**, *63*, 054102. [CrossRef]

22. Beretta, G.P. Nonlinear Extensions of Schrödinger–von Neumann Quantum Dynamics: A Set of Necessary Conditions for Compatibitliy with Thermodynamics. *Mod. Phys. Lett. A* **2005**, *20*, 977–984. [CrossRef]

23. Beretta, G.P. Nonlinear Quantum Evolution Equations to Model Irreversible Adiabatic Relaxation with Maximal Entropy Production and Other Nonunitary Processes. *Rep. Math. Phys.* **2009**, *64*, 139–168. [CrossRef]

24. Beretta, G.P. Maximum Entropy Production Rate in Quantum Thermodynamics. *J. Phys. Conf. Ser.* **2010**, *237*, 012004. [CrossRef]

25. Cano-Andrade, S.; Beretta, G.P.; von Spakovsky, M.R. Steepest-Entropy-Ascent Quantum Thermodynamic Modeling of Decoherence in Two Different Microscopic Composite Systems. *Phys. Rev. A* **2015**, *91*, 013848. [CrossRef]

26. Von Spakovsky, M.R.; Gemmer, J. Some Trends in Quantum Thermodynamics. *Entropy* **2014**, *16*, 3434–3470. [CrossRef]

27. Ray, R.K. Steepest Entropy Ascent Solution for a Continuous-Time Quantum Walker. *Phys. Rev. E* **2022**, *106*, 024115. [CrossRef]

28. Tabakin, F. Model Dynamics for Quantum Computing. *Ann. Phys.* **2017**, *383*, 33–78. [CrossRef]

29. Tabakin, F. Local Model Dynamics for Two Qubits. *Ann. Phys.* **2023**, *457*, 169408. [CrossRef]

30. Park, J.L. The Concept of Transition in Quantum Mechanics. *Found. Phys.* **1970**, *1*, 23–33. [CrossRef]

31. Ghirardi, G. Entanglement, Nonlocality, Superluminal Signaling and Cloning. In *Advances in Quantum Mechanics*; Bracken, P., Ed.; IntechOpen: London, UK, 2013; pp. 565–594. [CrossRef]

32. Wootters, W.K.; Zurek, W.H. A Single Quantum Cannot Be Cloned. *Nature* **1982**, *299*, 802–803. [CrossRef]

33. Dieks, D. Communication by EPR Devices. *Phys. Lett. A* **1982**, *92*, 271–272. [CrossRef]

34. Eberhard, P.H. Bell's Theorem and the Different Concepts of Locality. *Il Nuovo Cimento B* **1978**, *46*, 392–419. [CrossRef]

35. Simon, C.; Bužek, V.; Gisin, N. No-Signaling Condition and Quantum Dynamics. *Phys. Rev. Lett.* **2001**, *87*, 170405. [CrossRef]

36. Bóna, P. Comment on "No-Signaling Condition and Quantum Dynamics". *Phys. Rev. Lett.* **2003**, *90*, 208901. [CrossRef] [PubMed]

37. Simon, C.; Bužek, V.; Gisin, N. Simon, Bužek, and Gisin Reply. *Phys. Rev. Lett.* **2003**, *90*, 208902. [CrossRef]

38. Weinberg, S. Testing Quantum Mechanics. *Ann. Phys.* **1989**, *194*, 336–386. [CrossRef]

39. Gisin, N. Weinberg's Non-Linear Quantum Mechanics and Supraluminal Communications. *Phys. Lett. A* **1990**, *143*, 1–2. [CrossRef]

40. Polchinski, J. Weinberg's Nonlinear Quantum Mechanics and the Einstein-Podolsky-Rosen Paradox. *Phys. Rev. Lett.* **1991**, *66*, 397–400. [CrossRef]

41. Wódkiewicz, K.; Scully, M.O. Weinberg's Nonlinear Wave Mechanics. *Phys. Rev. A* **1990**, *42*, 5111–5116. [CrossRef]

42. Czachor, M. Nonlinear Schrödinger Equation and Two-Level Atoms. *Phys. Rev. A* **1996**, *53*, 1310–1315. [CrossRef]

43. Czachor, M. Mobility and Non-Separability. *Found. Phys. Lett.* **1991**, *4*, 351–361. [CrossRef]

44. Gisin, N.; Rigo, M. Relevant and Irrelevant Nonlinear Schrodinger Equations. *J. Phys. A Math. Gen.* **1995**, *28*, 7375–7390. [CrossRef]

45. Abrams, D.S.; Lloyd, S. Nonlinear Quantum Mechanics Implies Polynomial-Time Solution for NP-Complete and # P Problems. *Phys. Rev. Lett.* **1998**, *81*, 3992–3995. [CrossRef]

46. Ferrero, M.; Salgado, D.; Sánchez-Gómez, J.L. Nonlinear Quantum Evolution Does Not Imply Supraluminal Communication. *Phys. Rev. A* **2004**, *70*, 014101. [CrossRef]

47. Rembieliński, J.; Caban, P. Nonlinear Evolution and Signaling. *Phys. Rev. Res.* **2020**, *2*, 012027. [CrossRef]

48. Rembieliński, J.; Caban, P. Nonlinear Extension of the Quantum Dynamical Semigroup. *Quantum* **2021**, *5*, 420. [CrossRef]

49. Kaplan, D.E.; Rajendran, S. Causal Framework for Nonlinear Quantum Mechanics. *Phys. Rev. D* **2022**, *105*, 055002. [CrossRef]

50. Schrödinger, E. Probability Relations between Separated Systems. *Math. Proc. Camb. Philos. Soc.* **1936**, *32*, 446–452. [CrossRef]

51. Park, J.L. Nature of Quantum States. *Am. J. Phys.* **1968**, *36*, 211–226. [CrossRef]

52. Park, J.L. Thermodynamic Aspects of Schrödinger's Probability Relations. *Found. Phys.* **1988**, *18*, 225–244. [CrossRef]

53. Beretta, G.P. The Hatsopoulos–Gyftopoulos Resolution of the Schrödinger–Park Paradox about the Concept of "State" in Quantum Statistical Mechanics. *Mod. Phys. Lett. A* **2006**, *21*, 2799–2811. [CrossRef]

54. Beretta, G.P. Intrinsic Entropy and Intrinsic Irreversibility for a Single Isolated Constituent of Matter: Broader Kinematics and Generalized Nonlinear Dynamics. In *Frontiers of Nonequilibrium Statistical Physics*; Moore, G.T., Scully, M.O., Eds.; Springer: Boston, MA, USA, 1986; pp. 205–212. [CrossRef]

55. Park, J.L.; Simmons, R.F. The Knots of Quantum Thermodynamics. In *Old and New Questions in Physics, Cosmology, Philosophy, and Theoretical Biology: Essays in Honor of Wolfgang Yourgrau*; Springer: Boston, MA, USA, 1983; pp. 289–308. [CrossRef]

56. Hatsopoulos, G.N.; Beretta, G.P. Where Is the Entropy Challenge? *AIP Conf. Proc.* **2008**, *1033*, 34–54. [CrossRef]

57. Allahverdyan, A.E.; Balian, R.; Nieuwenhuizen, T.M. Maximal Work Extraction from Finite Quantum Systems. *Europhys. Lett.* **2004**, *67*, 565. [CrossRef]

58. Beretta, G.P. On the General Equation of Motion of Quantum Thermodynamics and the Distinction between Quantal and Nonquantal Uncertainties. *arXiv* **2005**, arXiv:quant-ph/0509116. [CrossRef]

59. Montañez-Barrera, J.A.; von Spakovsky, M.R.; Damian Ascencio, C.E.; Cano-Andrade, S. Decoherence Predictions in a Superconducting Quantum Processor Using the Steepest-Entropy-Ascent Quantum Thermodynamics Framework. *Phys. Rev. A* **2022**, *106*, 032426. [CrossRef]

60. Li, G.; von Spakovsky, M.R. Generalized Thermodynamic Relations for a System Experiencing Heat and Mass Diffusion in the Far-From-Equilibrium Realm Based on Steepest Entropy Ascent. *Phys. Rev. E* **2016**, *94*, 032117. [CrossRef]

61. Li, G.; von Spakovsky, M.R.; Hin, C. Steepest Entropy Ascent Quantum Thermodynamic Model of Electron and Phonon Transport. *Phys. Rev. B* **2018**, *97*, 024308. [CrossRef]

62. Yamada, R.; von Spakovsky, M.R.; Reynolds, W.T. Predicting Continuous and Discontinuous Phase Decompositions Using Steepest-Entropy-Ascent Quantum Thermodynamics. *Phys. Rev. E* **2019**, *99*, 052121. [CrossRef]

63. Montañez Barrera, J.A.; Damian-Ascencio, C.E.; von Spakovsky, M.R.; Cano-Andrade, S. Loss-of-Entanglement Prediction of a Controlled-Phase Gate in the Framework of Steepest-Entropy-Ascent Quantum Thermodynamics. *Phys. Rev. A* **2020**, *101*, 052336. [CrossRef]

64. von Neumann, J. *Mathematical Foundations of Quantum Mechanics*; Beyer, R.T., Translator; Princeton University Press: Princeton, NJ, USA, 2018. [CrossRef]

65. Beretta, G.P. The Fourth Law of Thermodynamics: Steepest Entropy Ascent. *Philos. Trans. R. Soc. A* **2020**, *378*, 20190168. [CrossRef]

66. Martyushev, L.M. Maximum Entropy Production Principle: History and Current Status. *Physics-Uspekhi* **2021**, *64*, 558–583. [CrossRef]

67. Swenson, R. A Grand Unified Theory for the Unification of Physics, Life, Information and Cognition (Mind). *Philos. Trans. R. Soc. A* **2023**, *381*, 20220277. [CrossRef]

68. Swenson, R. The Fourth Law of Thermodynamics (LMEP) and How It Makes the World. *Preprints* **2024**, 2024081109. Available online: https://www.preprints.org/manuscript/202408.1109/v1 (accessed on 19 September 2025).

69. Beretta, G.P. Steepest Entropy Ascent Model for Far-Nonequilibrium Thermodynamics: Unified Implementation of the Maximum Entropy Production Principle. *Phys. Rev. E* **2014**, *90*, 042113. [CrossRef] [PubMed]

70. Beretta, G.P. Time–Energy and Time–Entropy Uncertainty Relations in Nonequilibrium Quantum Thermodynamics under Steepest-Entropy-Ascent Nonlinear Master Equations. *Entropy* **2019**, *21*, 679. [CrossRef] [PubMed]

71. Murthy, C.; Babakhani, A.; Iniguez, F.; Srednicki, M.; Yunger Halpern, N. Non-Abelian Eigenstate Thermalization Hypothesis. *Phys. Rev. Lett.* **2023**, *130*, 140402. [CrossRef]

72. Simmons, R.F.; Park, J.L. On Completely Positive Maps in Generalized Quantum Dynamics. *Found. Phys.* **1981**, *11*, 47–55. [CrossRef]

73. Raggio, G.A.; Primas, H. Remarks on "On Completely Positive Maps in Generalized Quantum Dynamics". *Found. Phys.* **1982**, *12*, 433–435. [CrossRef]

74. Simmons, R.F.; Park, J.L. Another Look at Complete Positivity in Generalized Quantum Dynamics: Reply to Raggio and Primas. *Found. Phys.* **1982**, *12*, 437–439. [CrossRef]

75. Pechukas, P. Reduced Dynamics Need Not Be Completely Positive. *Phys. Rev. Lett.* **1994**, *73*, 1060–1062. [CrossRef]

76. Alicki, R. Comment on "Reduced Dynamics Need Not Be Completely Positive". *Phys. Rev. Lett.* **1995**, *75*, 3020–3020. [CrossRef] [PubMed]

77. Pechukas, P. Pechukas Replies. *Phys. Rev. Lett.* **1995**, *75*, 3021–3021. [CrossRef] [PubMed]

78. Rivas, Á.; Huelga, S.F.; Plenio, M.B. Entanglement and Non-Markovianity of Quantum Evolutions. *Phys. Rev. Lett.* **2010**, *105*, 050403. [CrossRef] [PubMed]

79. Jagadish, V.; Srikanth, R.; Petruccione, F. Measure of Positive and Not Completely Positive Single-Qubit Pauli Maps. *Phys. Rev. A* **2019**, *99*, 022321. [CrossRef]

80. Horodecki, R.; Horodecki, P.; Horodecki, M. Violating Bell Inequality by Mixed Spin-12 States: Necessary and Sufficient Condition. *Phys. Lett. A* **1995**, *200*, 340–344. [CrossRef]

81. Brüning, E.; Mäkelä, H.; Messina, A.; Petruccione, F. Parametrizations of Density Matrices. *J. Mod. Opt.* **2012**, *59*, 1–20. [CrossRef]

82. Byrd, M.S.; Khaneja, N. Characterization of the Positivity of the Density Matrix in Terms of the Coherence Vector Representation. *Phys. Rev. A* **2003**, *68*, 062322. [CrossRef]

83. Horodecki, R.; Horodecki, M. Information-Theoretic Aspects of Inseparability of Mixed States. *Phys. Rev. A* **1996**, *54*, 1838–1843. [CrossRef]

84. Lang, M.D.; Caves, C.M. Quantum Discord and the Geometry of Bell-Diagonal States. *Phys. Rev. Lett.* **2010**, *105*, 150501. [CrossRef]

85. Werner, R.F. Quantum States with Einstein-Podolsky-Rosen Correlations Admitting a Hidden-Variable Model. *Phys. Rev. A* **1989**, *40*, 4277–4281. [CrossRef]

86. Popescu, S.; Rohrlich, D. Causality and Nonlocality as Axioms for Quantum Mechanics. In *Causality and Locality in Modern Physics*; Hunter, G., Jeffers, S., Vigier, J.P., Eds.; Springer: Dordrecht, The Netherlands, 1998; pp. 383–389. [CrossRef]

87. Wolfram Research Inc. Mathematica Online. 2022. Available online: https://www.wolfram.com/mathematica (accessed on 20 July 2023).